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Efficacy of Variational Iteration Method for Nonlinear Heat Transfer Equations – Classical and Multistage Approach

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Abstract – *This is paper, we are study of solving nonlinear convective-radiative cooling equation by using the classical variational iteration method (VIM) and modified version called the multistage variational iteration method (MVIM). This method is based on the use of Lagrange multipliers for identification of optimal values of parameters in a functional. Furthermore, a theorem for the convergence of the MVIM is presented and comparison of the results obtained by VIM, MVIM and homotopy-perturbation method (HPM) with exact solutions.*

***Keywords**: Heat Transfer, Nonlinear Equation, Variational Iteration Method, Convergence*

I. Introduction

Heat transfer stills the important engineering problems [1]-[27]. Most of these problems cannot be analytically solving use traditional. Some of them are solved using numerical techniques and some are solved using the analytical method of perturbation. The exact analytical solutions or to solve the problem with the help of some suitable numerical technique. Fortunately, the present work shows that all the above models are exactly solvable in terms of algebraic function, Lumbert W function and hypergeometric function. Therefore, the researchers are looking for analytical or numerical of these problems. Most common methods used are variational iteration method (VIM) [21]-[22], Homtopy-perturbation method (HPM) [23]-[24], homotopy analysis method (HAM) [25]-[26], optimal homotopy asymptotic method (OHAM) [27].

The variational iteration method (VIM) was first proposed by Ji-Huan He for solving a wide range of problems whose mathematical models yield a differential equation or system of differential Eqs. [1], [2]. The idea of VIM is to construct an iteration method based on a correction functional that includes a generalized Lagrange multiplier.

The value of the multiplier is chosen using variational theory so that each iteration improves the accuracy of the solution. The initial approximation (trial function) usually includes unknown coefficients which can be determined to satisfy any boundary and initial conditions. The method usually gives rapidly convergent successive approximations to the exact solution if such a solution exists. VIM has been applied successfully to many different problems. For example, Wazwaz [8] used VIM to solve linear and nonlinear Schrodinger equations. Dehghan and Shakeri [9] applied VIM to solve the Cauchy reaction–diffusion problem.

Ravi Kanth and Arana [10] used VIM to solve treating nonlinear singular boundary value problems. Rezazadeh et al. [11] applied VIM to parametric oscillation of an eletrostatically actuated microbeam. Yang and Chen [12] used VIM to obtain Choice of an optimal initial solution for a wave equation. Torvattanabun and Koonprasert [14] applied VIM to convergence and solve a First-Order Linear System of PDEs with Constant Coefficients. Torvattanabun and Koonprasert [15] used VIM to solve Eighth-Order Boundary Value Problems.

Multistage variational iteration method (MVIM) was first introduced by Batiha et al.[17] on a class of nonlinear system of ordinary differential equations. This MVIM offers accurate solutions over a longer time frame (more stable) compared to the VIM. The distinctive strategy grants the iterative algorithm a time-marching scheme which significantly drives forward the convergence of the solutions precisely with great rapidity. The motivation of this paper is organized as follows: Section2 proposed method. Section3 the application to extend multistage variational iteration method to solve nonlinear convective-radiative cooling equation. Section4 shows an analysis of convergence on MVIM. Section5 presents the results obtained by the methods and compare the obtained results with MVIM, VIM and HPM. Section5 concluding on the method used.

II. Variational Iteration Method and Multistage Variational Iteration Method

II.1. Variational Iteration Method (VIM)

Consider the following differential equation:

$$Lu + Nu = g(t) \tag{1}$$

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where L is a linear operator, N is a nonlinear operator and $g(t)$ is a known real function. According to VIM [1]-[7], we can construct a correction functional, $u(t)$, as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\left(Lu_n(x,s) + N\tilde{u}_n(x,s) - g(x,s)\right)ds \quad (2)$$

where $\tilde{u}_n(t)$ is a correction functional but $\tilde{u}_n(x,t)$ is considered as a restricted variation [4]-[6], i.e. $\delta\tilde{u}_n(x,t) = 0$. The subscript n denotes the nth-order approximation. The optimal value of the general Lagrange multipliers λ [6] can be identified by using the stationary conditions of the variational theory.

II.2. Multistage Variational Iteration Method (MVIM)

For the correction function (2), where t^* will change or in fact increase according to the designated time-step in each iteration computation. This methodology was inspired by various researches [17]-[19]:

$$u_{n+1}(x,t) = u_n(x,t) + \int_{t^*}^t \lambda\left(Lu_n(x,s) + N\tilde{u}_n(x,s) - g(x,s)\right)ds \quad (3)$$

Notice that this strategy gives a new construction of the correction functional (3) with a variable t^*.

III. Application of Multistage Variational Iteration Method

III.1. Cooling of a Lumped System with Variable Specific Heat

Consider the problem of combined convective-radiative cooling of a lumped system [20]. Suppose the system have volume V, surface area A, density ρ, specific heat c, emissivity E and initial temperature T_i, at time $t=0$. The system is exposed to an environment heat transfer with the coefficient of h and the temperature T_a. Assume that the specific heat c is a linear function temperature of the form:

$$c = c_a\left[1+\beta(T-T_a)\right] \quad (4)$$

where c_a is the specific heat, at temperature T_a and β is a constant.

The cooling equation and the initial condition are as follows:

$$\rho Vc\frac{dT}{dt} + hA(T-T_a) = 0, \quad t=0,\ T=T_i \quad (5)$$

To solve the equation, we do changes of parameters:

$$\theta = \frac{T-T_a}{T_i-T_a},\ \tau = \frac{t}{\rho Vc_a/hA},\ \varepsilon = \beta(T-T_a) \quad (6)$$

After parameter change, the heat transfer equation will result the following:

$$(1+\varepsilon\theta)\frac{d\theta}{d\tau} + \theta = 0, \quad \tau = 0, \theta = 1 \quad (7)$$

III.2. MVIM

For solving the problem (7) by MVIM, then we construct the following a correction functional for $\theta(\tau)$:

$$\theta_{n+1}(\tau) = \theta_n(\tau) + \int_{\tau^*}^{\tau} \lambda(s)\left[(1+\varepsilon\theta_n(s))\frac{d\tilde{\theta}_n(s)}{ds} + \theta_n(s)\right]ds \quad (8)$$

where λ is general Lagrange multiplier, n denotes the nth approximation, and $\tilde{\theta}_n(\tau)$ denote restricted variations, i.e. $\delta\tilde{\theta}_n(\tau) = 0$. The variation of (8) is then:

$$\delta\theta_{n+1}(\tau) = \delta\theta_n(\tau) + \delta\int_{\tau^*}^{\tau} \lambda(s)\left[(1+\varepsilon\theta_n(s))\frac{d\tilde{\theta}_n(s)}{ds} + \theta_n(s)\right]ds \quad (9)$$

By using integration by parts, we have the stationary conditions are as follows:

$$\lambda'(s) - \lambda(s) = 0,\ 1+\lambda(s) = 0 \text{ then } \lambda(s) = -e^{s-\tau}$$

Therefore, from(8) the following formula for computing $\theta_m(\tau)$ may be obtained:

$$\theta_{n+1}(\tau) = \theta_n(\tau) - \int_{\tau^*}^{\tau} e^{s-\tau}\left[(1+\varepsilon\theta_n(s))\frac{d\theta_n(s)}{ds} + \theta_n(s)\right]ds \quad (10)$$

IV. Analysis of Convergence

Theorem: Let $\theta(\tau), \theta_i(\tau) \in C^1[0,l], i = 0,1,\ldots$.

The sequence defined by (10) with $\theta_0(\tau) = e^{-\tau}$ converges to $\theta(\tau)$, the exact solution of

(7). *Proof.* Since $\theta(\tau)$ is the exact solution of (7), it is obvious that:

$$\theta(\tau)=\theta(\tau)-\int_{\tau^*}^{\tau} e^{s-\tau}\left[\left(1+\varepsilon\theta(s)\right)\frac{d\theta(s)}{ds}+\theta(s)\right]ds \quad (11)$$

From (10) and (11), the error function is:

$$\begin{aligned}E_{m+1}(\tau)&=E_m(\tau)+\\&-\int_{\tau^*}^{\tau} e^{s-\tau}\left[\left(1+\varepsilon E_m(s)\right)\frac{dE_m(s)}{ds}+E_m(s)\right]ds=\\&=E_m(\tau)-\int_{\tau^*}^{\tau} e^{\tau-s}\frac{dE_m(s)}{ds}+\\&+e^{\tau-s}\varepsilon E_m(s)\frac{dE_m(s)}{ds}+e^{\tau-s}E_m(s)ds\end{aligned} \quad (12)$$

By using integration by parts and the fact that $E_m(0)=0, m=0,1,\ldots$ we conclude that:

$$\begin{aligned}E_{m+1}(\tau)&=-2\int_{\tau^*}^{\tau} e^{\tau-s}E_m(s)\,ds+\\&-\int_{\tau^*}^{\tau} e^{\tau-s}E_m(s)\frac{dE_m(s)}{ds}ds\end{aligned} \quad (13)$$

Therefore:

$$\begin{aligned}\left|E_{m+1}(\tau)\right|&\le 2\int_{\tau^*}^{\tau}\left|e^{\tau-s}\right|\left|E_m(s)\right|ds+\\&+|\varepsilon|\int_{\tau^*}^{\tau}\left|e^{\tau-s}\right|\left|E_m(s)\right|\left|\frac{dE_m(s)}{ds}\right|ds\end{aligned} \quad (14)$$

Since $s\le\tau\le l$, then $\left|e^{\tau-s}\right|\le e^{2l}$ and let $a=\max\left(2e^{2l},|\varepsilon|e^{2l}\right)$. Hence, from (14) we obtain:

$$\begin{aligned}\left|E_{m+1}(\tau)\right|&=a\int_{\tau^*}^{\tau}\left|E_m(s)\right|ds+\\&+a\int_{\tau^*}^{\tau}\left|E_m(s)\right|\left|\frac{dE_m(s)}{ds}\right|ds\end{aligned} \quad (15)$$

Now we proceed to obtain:

$$\left|E_1(\tau)\right|=a\left(\int_{\tau^*}^{\tau}\left|E_0(s)\right|ds+\int_{\tau^*}^{\tau}\left|E_0(s)\right|\left|\frac{dE_0(s)}{ds}\right|ds\right)\le$$

$$a\left(\max_{s\in[\tau^*,\tau]}\left|E_0(s)\right|\int_{\tau^*}^{\tau}ds+\max_{s\in[\tau^*,\tau]}\left|E_0(s)\right|\max_{s\in[\tau^*,\tau]}\left|\frac{dE_0(s)}{ds}\right|\int_{\tau^*}^{\tau}ds\right)$$

Let $c_0=\max\left(1,\max_{s\in\left[\tau^*,\tau\right]}\left|\frac{dE_0(s)}{ds}\right|\right)$, hence:

$$\left|E_1(s)\right|\le 2ac_0\int_{\tau^*}^{\tau}ds\le 2ac_0\left(\tau-\tau^*\right) \quad (16)$$

Let $M_m=2ac_m$, proceed as follows:

$$\begin{aligned}&\left|E_1(s)\right|\le M_0\left(\tau-\tau^*\right)\\&\left|E_2(\tau)\right|=a\left(\int_{\tau^*}^{\tau}\left|E_1(s)\right|ds+\int_{\tau^*}^{\tau}\left|E_1(s)\right|\left|\frac{dE_1(s)}{ds}\right|ds\right)\\&\le a\left(\int_{\tau^*}^{\tau}\left|E_1(s)\right|ds+\max_{s\in\left[\tau^*,\tau\right]}\left|\frac{dE_1(s)}{ds}\right|\int_{\tau^*}^{\tau}\left|E_1(s)\right|ds\right)\\&\le M_1\int_{\tau^*}^{\tau}\left|E_1(s)\right|ds\le M_1M_0\max_{s\in\left[\tau^*,\tau\right]}\left|E_0(s)\right|\frac{\left(\tau-\tau^*\right)^2}{2}\end{aligned} \quad (17)$$

$$\begin{aligned}\left|E_3(\tau)\right|&\le M_2\int_{\tau^*}^{\tau}\left|E_2(s)\right|ds\\&\le M_2M_1M_0\max_{s\in\left[\tau^*,\tau\right]}\left|E_0(s)\right|\frac{\left(\tau-\tau^*\right)^3}{3!}\\&\vdots\\\left|E_m(\tau)\right|&\le M_{m-1}\int_{\tau^*}^{\tau}\left|E_{m-1}(s)\right|ds\\&\le M_m\ldots M_0\max_{s\in\left[\tau^*,\tau\right]}\left|E_0(s)\right|\frac{\left(\tau-\tau^*\right)^m}{m!}\end{aligned} \quad (18)$$

Let $M=\max_{i=0}^{m-1}M_i$, hence:

$$\left|E_m(\tau)\right|\le\max_{s\in\left[\tau^*,\tau\right]}\left|E_0(s)\right|M^m\frac{\left(\tau-\tau^*\right)^m}{m!} \quad (19)$$

We have:

$$\begin{aligned}&\max_{s\in\left[\tau^*,\tau\right]}\left|E_0(s)\right|M^m\frac{\left(\tau-\tau^*\right)^m}{m!}\\&\le\max_{s\in\left[\tau^*,\tau\right]}\left|E_0(s)\right|M^m\frac{\left(l-\tau^*\right)^m}{m!}\to 0\end{aligned} \quad (20)$$

as $m\to\infty$, hence the functional sequence $\left\{\theta_m(\tau)\right\}_{m=1}^{\infty}$ convergence to $\theta(\tau)$.

V. Results and Discussions

The computational algorithm is written in the Maple software for the iterations to run its course. Eqs. (10) illustrates the approximate solutions for two iteration step using a time-step of τ^*.

$$\begin{aligned}
\theta_0(\tau) &= e^{-\tau} \\
\theta_1(\tau) &= e^{-\tau} + \varepsilon\left(e^{-\tau-\tau^*} - e^{-2\tau}\right) \\
\theta_2(\tau) &= e^{-\tau} + \varepsilon\left(e^{-\tau-\tau^*} - e^{-2\tau}\right) + \\
&+ \varepsilon^2\left(\frac{1}{2}e^{-\tau-2\tau^*} - 2e^{-2\tau-\tau^*} + \frac{3}{2}e^{-3\tau}\right) \\
&+ \varepsilon^3\left(\frac{3}{2}e^{-3\tau-\tau^*} + \frac{1}{6}e^{-\tau-3\tau^*} - \frac{2}{3}e^{-4\tau} - e^{-2\tau-2\tau^*}\right)
\end{aligned} \tag{21}$$

We obtain VIM see [21]:

$$\begin{aligned}
\theta_2(\tau) &= e^{-\tau} + \varepsilon\left(e^{-\tau} - e^{-2\tau}\right) + \\
&+ \varepsilon^2\left(\frac{1}{2}e^{-\tau} - 2e^{-2\tau} + \frac{3}{2}e^{-3\tau}\right) + \\
&+ \varepsilon^3\left(\frac{3}{2}e^{-3\tau} + \frac{1}{6}e^{-\tau} - \frac{2}{3}e^{-4\tau} - e^{-2\tau}\right)
\end{aligned} \tag{22}$$

Obtain HPM see [23]:

$$\begin{aligned}
\theta(\tau) &= e^{-\tau} + \varepsilon\left(e^{-\tau} - e^{-2\tau}\right) + \\
&+ \varepsilon^2\left(\frac{1}{2}e^{-\tau} - 2e^{-2\tau} + \frac{3}{2}e^{-3\tau}\right)
\end{aligned} \tag{23}$$

We used time-step of $\Delta t = 0.001$ for MVIM. We found that, overall, MVIM gave a much better performance in approximating solutions compared to VIM [21]and HPM [23], shown in table.

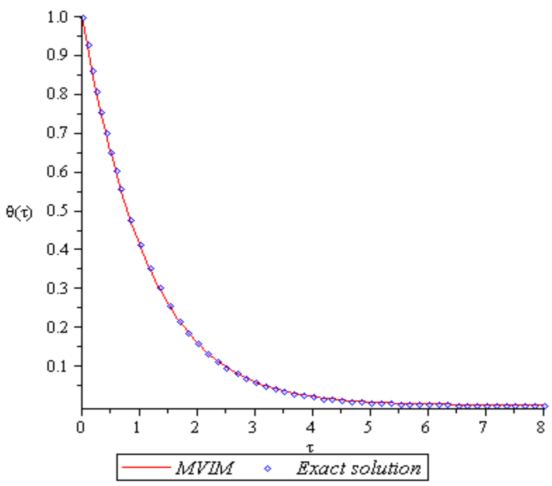

Fig. 1. The comparison of the results the MVIM of time-step $\Delta t = 0.001$ with exact solution at $\tau = 0.2$

TABLE I
THE RESULTS OF VIM, HPM, MVIM AND THEIR ERRORS AT $\tau = 0.5$

ε	MVIM	Error of MVIM	Error of VIM	Error of HPM
0	0.6065306598	0.238E-2	0.238E-2	0.238E-2
0.1	0.6294547749	0.261E-4	0.331E-4	0.3021E-2
0.2	0.6502858407	0.164E-3	0.280E-3	0.180E-1
0.3	0.6688893541	0.820E-3	0.987E-3	0.269E-1
0.4	0.6851308122	0.220E-2	0.241E-2	0.475E-1
0.5	0.6988757117	0.459E-2	0.485E-2	0.737E-1
0.6	0.7099895493	0.825E-2	0.855E-2	0.10544
0.7	0.7183378220	0.134E-1	0.137E-1	0.14247
0.8	0.7237860267	0.204E-1	0.208E-1	0.18471
0.9	0.7261916600	0.294E-1	0.298E-1	0.23205
1.0	0.7254421900	0.408E-1	0.411E-1	0.28440

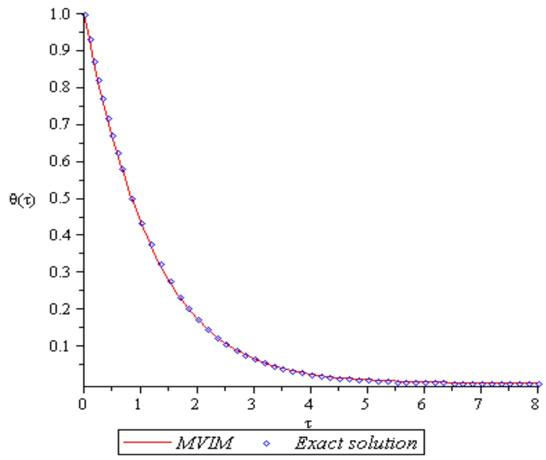

Fig. 2. The comparison of the results the MVIM of time-step $\Delta t = 0.001$ with exact solution at $\tau = 0.3$

VI. Conclusion

We have proved a convergence theorem for multistage variational iteration method. Comparisons with the exact solutions. We conclude that MVIM is an effective analytical tool for solving cooling of a lumped system with variable specific heat.

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