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A Research Note on a Solution of Stefan Problem with Fractional Time and Space Derivatives

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Abstract – *Stefan problem based on nonlocal heat conduction equation with fractional-time derivatives has been solved. The approach suggests a generalized time-dependent Stefan boundary condition defined by expression $\vartheta(\alpha,\beta,\tau)=\sigma(\alpha,\beta)\cdot\tau^{\alpha/\beta}$. The solution developed shows that the phase change boundary co-ordinate ϑ depends on time τ and the parameters α $(0<\alpha\leq 1)$ and β $(1<\beta\leq 2)$ which are the fractional orders with respect to the time and the space co-ordinate , respectively. A practical example with ice-water system was used to exemplify the solution with both α and β near to $\alpha=1$ and $\beta=1$ (the classical problem).*

***Keywords**: Fractional Derivative, Nonlocal Heat Conduction Equation, Stefan Problem*

Nomenclature

$a_0 = x_0^2 / t_0$	Heat diffusivity	$(\mathrm{m^2/s})$
c	Water temperature ($c>0$)	$(^{\circ}\mathrm{C})$
c_1	Ice temperature ($c_1<0$)	$(^{\circ}\mathrm{C})$
$Q=\rho_{\mathrm{ice}} w a_0$	Rate of heat release during the of phase change	$(\mathrm{kJ/m\,s})$
t_0	Characteristic time	
x_0	Characteristic length scale	(m)
w	Latent heat of the ice	$(\mathrm{kJ/kg})$

Greek symbols

α	Fractional time order, $0<\alpha\leq 1$	$(-)$
β	Space fractional order, $1<\beta\leq 2$	$(-)$
γ	$=\beta/2$	$(-)$
$a_{1(2)}=\dfrac{\lambda_{1(2)}}{c_{1(2)p}\rho_{1(2)}}$	Thermal diffusivity	$(\mathrm{m^2/s})$
λ	Heat conduction coefficient	$(\mathrm{W/mK})$
c_p	Specific isobaric heat capacity	$(\mathrm{kJ/kg})$
ρ	density	$(\mathrm{kg/m^3})$
σ	Is a time-independent pre-factor	$(-)$
$\tau=t/t_0$	Dimensionless time	$(-)$
$\xi=x/x_0$	Dimensionless space co-ordinate	$(-)$
$\vartheta(\tau)$	Dimensionless interphase co-ordinate	$(-)$

I. Introduction

Stefan problem [1], [2] is frequently encountered in modelling articles oriented to phase –change problems and many substantial improvements in the solutions have been developed. [3],[4],[5]. Mathematically, the Stefan problem is a typical non-linear problem allowing a generalization it allows to generalize due to moving phase-change boundary. In context, many interesting features of the moving interphase boundary have been analyzed elsewhere [6].

A new branch in the Stefan problem family is the motion of the interphase boundary described as a non-local process and represented by derivatives of fractional orders encountering either hereditary time processes or some fractal space structures [7],[8],[9],[10],[11].

The present work address the memory effects in the motion of the phase-change boundary ϑ expressed as function of the time τ and the space coordinate and, the fractional orders $0<\alpha\leq 1$, $1<\beta\leq 2$, as well.

II. Non-Local Stefan Problem

Consider a Stefan problem relevant to an ice-water system [12]. The heat balance equations for both phases are expressed by fractional derivative with respect to the time and the space co-ordinates Equations in Stefan

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problem in derivative of fractional order have the form:

$$\frac{\partial^\alpha T_1(\xi,\tau)}{\partial\tau^\alpha}-D_1\frac{\partial^\beta T_1(\xi,\tau)}{\partial\xi^\beta}=0,\ \ 0<\xi<\vartheta(\tau) \quad (1)$$

$$\frac{\partial^\alpha T_2(\xi,\tau)}{\partial\tau^\alpha}-D_2\frac{\partial^\beta T_2(\xi,\tau)}{\partial\xi^\beta}=0,\ \ \vartheta(\tau)<\xi<\infty \quad (2)$$

where:

$$\frac{\partial^\alpha \mathrm{T}(\xi,\tau)}{\partial\tau^\alpha}=\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial\tau}\int_0^\tau\frac{\mathrm{T}(\xi,z)}{(\tau-z)^\alpha}dz-\frac{T(\xi,0)}{\Gamma(1-\alpha)\tau^\alpha} \quad (3)$$

$$\frac{\partial^\beta \mathrm{T}(\xi,\tau)}{\partial\xi^\beta}=\frac{1}{2\Gamma(2-\beta)cos\left(\frac{\pi}{2}(2-\beta)\right)}\times \times\frac{\partial^2}{\partial\xi^2}\int_0^\infty\frac{\mathrm{T}(\xi',\tau)}{|\xi-\xi'|^{\beta-1}}d\xi' \quad (4)$$

Here $D_{1(2)}=a_{1(2)}t_0/x_0^2$ is a dimensionless thermal diffusivity. The Caputo derivative in eq. (3) accounts nonlocal time effects, while Riesz derivative in eq. (4) refers to spatial nonlocal effects.

The boundary conditions to the above set equations ((1), (2)) are:

$$T_1(\xi=0,\tau)=c_1,\ T_2(\xi,\tau=0)=c \quad (5)$$

$$T_1(\xi,\tau)=T_2(\xi,\tau)=0\ \ for\ \ \xi=\vartheta(\tau) \quad (6)$$

$$\lambda_1\frac{\partial^\gamma T_1(\xi,\tau)}{\partial\xi^\gamma}-\lambda_2\frac{\partial^\gamma T_2(\xi,\tau)}{\partial\xi^\gamma}=Q\frac{\partial^\mu\vartheta(\tau)}{\partial\tau^\mu}\ \ for\ \ \xi=\vartheta(\tau) \quad (7)$$

III. Solution

The solution in a semi-infinite space can be expressed as [13]:

$$T(\xi,\tau)=\frac{2}{\pi}\int_0^\infty dk\int_0^\infty d\xi'\psi(\xi')sin(k\xi)sin(k\xi')E_{\alpha,1}\left(-Dk^\beta\tau^\alpha\right)+ +\frac{2D}{\pi}\int_0^\infty dk\int_0^\tau d\tau'\frac{sin(k\xi)}{k^{1-\beta}}\mu(0,\tau-\tau')\cdot\tau'^{\alpha-1}E_{\alpha,\alpha}\left(-Dk^\beta\tau'^\alpha\right) \quad (8a)$$

with an initial condition:

$$T(\xi,\tau=0)=\psi(\xi) \quad (8b)$$

and:

$$T(\xi=0,\tau)=\mu(\tau) \quad (8c)$$

as a boundary condition.

Here, $E_{\alpha,\nu}\left(-z^\alpha\right)=\sum_{n=0}^\infty(-1)^n\frac{z^{\alpha n}}{\Gamma(\alpha n+\nu)}$ is the Mittag-Leffler function [5]. When $\psi(\xi)=B$ and $\mu(\tau)=A$ - are constants, then the solution of (8(a),(b),(c)) is:

$$T(\xi,\tau)=A+\frac{2}{\pi}(B-A)\int_0^\infty dk\frac{sin(k\xi)}{k}E_{\alpha,1}\left(-Dk^\beta\tau^\alpha\right) \quad (9)$$

For $\alpha=1$, the solution (9) coincides with the classical solution of the Stefan problem [12].

From the solution (9) we can express consequently the solutions of the equations (1) and (2), namely:

$$T_1(\xi,\tau)=A_1+ +(B_1-A_1)\frac{2}{\pi}\int_0^\infty dz\frac{sin(z)}{z}E_{\alpha,1}\left(-D_1\frac{\tau^\alpha}{\xi^\beta}z^\beta\right) \quad (10)$$
$$for\ \ 0<\xi<\vartheta(\tau)$$

and:

$$T_2(\xi,\tau)=A_2+ +(B_2-A_2)\frac{2}{\pi}\int_0^\infty dz\frac{sin(z)}{z}E_{\alpha,1}\left(-D_2\frac{\tau^\alpha}{\xi^\beta}z^\beta\right) \quad (11)$$
$$for\ \ \vartheta(\tau)<\xi<\infty$$

From the boundary conditions (5) we get: $A_1=c_1$, $B_2=c$. Further, the boundary condition (6) results in the following expressions for calculating A_2 and B_1, namely:

$$c_1+(B_1-c_1)\frac{2}{\pi}\int_0^\infty dz\frac{sin(z)}{z}E_{\alpha,1}\left(-D_1\frac{\tau^\alpha}{\vartheta^\beta}z^\beta\right)=0 \quad (12)$$

$$A_2+(c-A_2)\frac{2}{\pi}\int_0^\infty dz\frac{sin(z)}{z}E_{\alpha,1}\left(-D_2\frac{\tau^\alpha}{\vartheta^\beta}z^\beta\right)=0 \quad (13)$$

The expressions (12) and (13) have to be satisfied for any time τ.

Moreover, the following two- phase boundary coordinate relation has to be satisfied:

$$\vartheta(\tau)=\sigma\cdot\tau^{\alpha/\beta} \quad (14)$$

with a time-independent pre-factor σ, that has to be defined.

The relation (14) defines the law of the interface motion and is, in fact, a generalization of the well-known relationship corresponding to the classical case (with $\alpha=1$ and $\beta=2$).

Substituting (14) into (12) and (13) we get the following expressions for A_2, B_1, namely:

$$B_1=-c_1\frac{\left(1-F_{\alpha,\beta}\left(D_1/\sigma^{\beta}\right)\right)}{F_{\alpha,\beta}\left(D_1/\sigma^{\beta}\right)} \tag{15a}$$

$$A_2=-c\frac{F_{\alpha,\varphi}\left(D_2/\sigma^{\beta}\right)}{1-F_{\alpha,\beta}\left(D_2/\sigma^{\beta}\right)} \tag{15b}$$

Here:

$$F_{\alpha,\beta}(x)=\frac{2}{\pi}\int_0^{\infty}dz\frac{sin(z)}{z}E_{\alpha,1}\left(-xz^{\beta}\right) \tag{16a}$$

$$\begin{gathered}K_{\alpha,\beta}(x)=\frac{1}{\pi cos\left(\frac{\pi}{2}(1-\gamma)\right)}\times\\ \times\int_0^{\infty}\frac{dz}{z^{1-\gamma}}\begin{pmatrix}cos\left(z+\frac{\pi}{2}(1-\gamma)\right)+\\ +U_{1-\gamma}(2z,0)\end{pmatrix}E_{\alpha,1}\left(-xz^{\beta}\right)\end{gathered} \tag{16b}$$

and $U_{\mu}(x,y)$ is the Lommel function of two arguments.

Then, the solutions of the equation take the forms:

$$T_1(\xi,\tau)=c_1\begin{pmatrix}1-\frac{1}{F_{\alpha,\beta}\left(D_1/\sigma^{\beta}\right)}\times\\ \times\frac{2}{\pi}\int_0^{\infty}dz\frac{sin\,z}{z}E_{\alpha,1}\left(-D_1\frac{\tau^{\alpha}}{\xi^{\beta}}z^{\beta}\right)\end{pmatrix} \tag{17a}$$

$$for\ \ 0<\xi<\vartheta(\tau)$$

$$\begin{gathered}T_2(\xi,\tau)=c\frac{F_{\alpha,\beta}\left(D_2/\sigma^{\beta}\right)}{1-F_{\alpha,\beta}\left(D_2/\sigma^{\beta}\right)}\times\\ \times\left(\frac{2}{\pi}\int_0^{\infty}dz\frac{sin(z)}{z}E_{\alpha,1}\left(-D_2\frac{\tau^{\alpha}}{\xi^{\beta}}z^{\beta}\right)-1\right)\end{gathered} \tag{17b}$$

$$for\ \ \vartheta(\tau)<\xi<\infty$$

Substituting the solutions (17a) and (17b) into the generalized Stefan condition (7) we get an equation defining the unknown pre-factor σ, namely:

$$\begin{gathered}\frac{c_1\lambda_1K_{\alpha,\beta}\left(D_1/\sigma^{\beta}\right)}{F_{\alpha,\beta}\left(D_1/\sigma^{\beta}\right)}+\frac{c\lambda_2K_{\alpha,\beta}\left(D_2/\sigma^{\beta}\right)}{\left(1-F_{\alpha,\beta}\left(D_2/\sigma^{\beta}\right)\right)}=\\ =-\frac{\Gamma(1+\alpha/\beta)}{\Gamma(1-\gamma\alpha/\beta)}Q\sigma^{1+\gamma}\end{gathered} \tag{18}$$

The expression (18) defines σ and generalizes traditional equation obtained for values $\alpha=1$, $\beta=2$ in (17a).

From (18) it follows that σ depends on parameters α and β, and the expression (14) can be expressed as:

$$\vartheta(\tau)=\sigma(\alpha,\beta)\tau^{\alpha/\beta} \tag{19}$$

It is important to note that the order μ of fractional derivative $Q\frac{\partial^{\mu}\vartheta(\tau)}{\partial\tau^{\mu}}$ in the right side of the Stefan condition (7) is determined from the condition that σ **is time-independent** equals $\mu=\frac{\alpha}{\beta}(1+\gamma)$.

IV. Example with an Ice-Water System

It is interesting to clarify the effects of the fractional orders (parameters) α and β on the function $\vartheta(\tau)$ considering a particular physical systems, ice-water for example with the following physical data:

- Setting, for instance, the water temperature equal to zero (i.e. c=0) and the ice temperature $c_1=-5^0$ C.
- The ice density $\rho_{ice}=917.4\,\text{kg}/\text{m}^3$, thermal conductivity $\lambda_{ice}=2.24W\,(\text{mK})$, $w_{ice}=330\,\text{kJ/kg}$ and a thermal diffusivity $a_{ice}=1.2\times10^{-6}\,\text{m}^2/\text{s}$.
- The cartelistic time is taken $t_0=1\text{h}$, for example, while the process length scale becomes $x_0=\sqrt{at_0}=6.6$ cm.

The variations of $\vartheta(\tau)$ as effects of with parameters α and β are shown in Fig. 1.

V. Conclusion

The solution developed demonstrates that the non-locality in time and space affects the position of the interphase boundary in the Stefan problem.

The non-local effects in time results is increased space co-ordinate of the interface, wile the non-locality with the respect to the space has the opposite effect. In accordance with the solution developed there is arrange near to $\alpha=1$ and $\beta=1$ where the function $\vartheta(\alpha,\beta,\tau)$ increases rapidly and cannot be controlled.

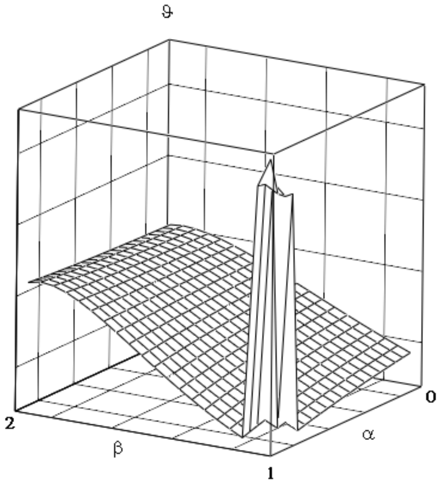

Fig. 1. The dimensionless interphase boundary $\vartheta = \vartheta(\alpha, \beta, \tau)$ as a function of parameters α and β and $\tau = 20$

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