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Approximate Solution of Fractional Diffusion Equation – Revisited

S. Das

Abstract – *The article presents the approximate analytical solution of a fractional diffusion equation with the help of powerful mathematical tool viz., variational iteration method. Fractional diffusion equation has special importance in engineering and science, and constitutes a good model for many systems in various fields. By using initial value, the explicit solution of the equation has been derived, which accelerate the rapid convergence of the series solution. The striking features of the article are successful presentation of sub-diffusion characteristics of the probability density function through numerical computation due to the presence of reaction term for various fractional Brownian motions and for different specified values of the considered parameters.*

***Keywords**: Fractional Derivative, Diffusion Equation, Initial Value, Sub-Diffusion, Reaction Term, Variational Iteration Method*

I. Introduction

Nowadays fractional differential equations (FDEs) have been focused due to their frequent appearances in various applications in fluid mechanics, viscoelasticity, biology, physics, electrical network, control theory of dynamical systems, chemical physics, optics and signal processing, which can be successfully modelled by linear and non-linear fractional order differential equations. The important reason of gaining attention towards the research in the area of fractional calculus is due to the fact that fractional order system response ultimately converges to the integer order system response. Oldham and Spanier [1] have played the key role for the development of the subject. Several fundamental works solving fractional differential equations had been done by Miller and Rose [2], Podlubny [3], Diethelm and Ford [4], Diethelm [5], etc.

Recently, application has been included to solve various classes of nonlinear fractional differential equations numerically. It is often seen that random fractal structures exhibit many anomalous features due to spatial complexities of the substrate which imposes geometrical constraints. These constraints may also be seen as temporal correlations existing on all time scales.

In case of diffusion, for instance, these correlations lead to an anomalous behaviour where the mean-square displacement of a Brownian particle for time fractional derivative is $X^2 = \left\langle x^2(t) \right\rangle > \approx t^{\alpha}$, where $0 < \alpha < 1$ is the anomalous diffusion exponent.

Variational iteration method (VIM) is one of the powerful methods by which a large variety of linear and nonlinear problems are solved with approximations converging rapidly to the exact and appropriate analytical solutions.

The variational iteration method was first proposed by Chinese Mathematician J. H. He ([6]–[10]) and was successfully applied to solve nonlinear systems of PDE's and nonlinear differential equations of fractional order by Shawagfeh [11], Diethelm and Ford [12], Momani and Odibat [13], etc.

Diffusion equations are extensively studied by the researchers across the world due to their tremendous applications in science and engineering. But when equations involve the nonlinear terms and also fractional order derivatives, the study takes different dimension and becomes very much challenging to the researchers.

Recently, the anomalous behaviors of the nonlinear fractional order diffusion equation in the form of sub- and super-diffusion due to the presence of different type of reaction term have been shown in the article of Das et al. [14].

In another recent article of Das et al. [15], the approximate analytical solutions of fractional order nonlinear diffusion equation in the presence of an absorbent term and a linear external force have been studied using another powerful mathematical tool like Homotopy Perturbation Method.

In general it is very difficult to get even an approximate solution of a fractional order nonlinear equation.

In this article the author has made a sincere endeavor to get the approximate solution of the following fractional diffusion equation in presence of reaction term:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial}{\partial x}\left(u^n \frac{\partial u}{\partial x}\right) - \int_0^t a(t-\xi)\, u(x,\xi)\, d\xi, \quad o < \alpha \le 1 \tag{1}$$

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with:

$$u(x,0)=x^k \tag{2}$$

where $a(t)=a\,\frac{t^{\beta-1}}{\Gamma(\beta)}$, $a>0, \beta>0$, is the time dependent absorbent term.

The salient feature of the article is to solve the equation (1) under the complicated initial condition (2). To the best of author's knowledge, the equation (1) has not yet been studied by any researcher under the complicated initial condition $u(x,0)=x^k$.

Effects of damping due to the presence of reaction term in order to get sub-diffusion of the nonlinear system are elegantly studied through numerical computation for different values of the parameters of physical interest and are presented graphically.

II. Solution of the Problem

The Eq (1) can be re-written as:

$$\frac{\partial u(x,t)}{\partial t}=\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}\left[\frac{\partial}{\partial x}\left(u^n\frac{\partial u}{\partial x}\right)-\int_0^t a(t-\xi)u(x,\xi)\,d\xi\right] \tag{3}$$

According to the VIM, the correction functional in t -direction can be written as:

$$\begin{aligned}&u_{n+1}(x,t)=u_n(x,t)+\\&+\int_0^t\lambda(\xi)\left[\begin{array}{l}\frac{\partial u_n(x,\xi)}{\partial\xi}-\frac{\partial^{1-\alpha}}{\partial\xi^{1-\alpha}}\\ \left\{\begin{array}{l}\frac{\partial}{\partial x}\left((\tilde{u}_n(x,\xi))^n\frac{\partial}{\partial x}\tilde{u}_n(x,\xi)\right)+\\ -\int_0^\xi a(\xi-\eta)\tilde{u}_n(x,\eta)\,d\eta\end{array}\right\}\end{array}\right]d\xi\end{aligned} \tag{4a}$$

The function $\tilde{u}_n$ is a restricted variation, which means $\delta\tilde{u}_n=0$.

The successive approximation $u_{n+1}(x,t)$, $n\geq 0$ of the solution $u(x,t)$ will be readily obtained upon using Lagrangian multiplier, and by using any selective function u_0. The initial value $u(x,0)$ is usually used for selecting the zero-th approximation u_0. To find the optimal value of $\lambda(\xi)$, we have:

$$\begin{aligned}&\delta u_{n+1}(x,t)=\delta\ \ u_n(x,t)+\\&+\delta\int_0^t\lambda(\xi)\frac{\partial u_n(x,\xi)}{\partial\xi}d\xi\ =0\end{aligned} \tag{4b}$$

This yields the stationary conditions:

$$\lambda'(\xi)=0 \text{ and } 1+\lambda(\xi)=0 \tag{4c}$$

Thus:

$$\lambda(\xi)=-1 \tag{5}$$

Substituting the value of λ in equation (4a), we get:

$$\begin{aligned}&u_{n+1}(x,t)=u_n(x,t)+\\&-\int_0^t\left[\begin{array}{l}\frac{\partial u_n(x,\xi)}{\partial\xi}+\\ -\frac{\partial^{1-\alpha}}{\partial\xi^{1-\alpha}}\left\{\begin{array}{l}\frac{\partial}{\partial x}\left((u_n(x,\xi))^n\frac{\partial u_n(x,\xi)}{\partial x}\right)+\\ -\int_0^\xi a(t-\eta)u_n(x,\eta)\,d\eta\end{array}\right\}\end{array}\right]d\xi\end{aligned} \tag{6}$$

Beginning with the initial approximation $u_0(x,t)=u(x,0)=x^k$ the successive approximations are obtained as:

$$u_1(x,t)=x^k+k(kn+k-1)x^{kn+k-2}\frac{t^\alpha}{\Gamma(\alpha+1)}-ax^k\frac{t^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)} \tag{7}$$

$$\begin{aligned}u_2(x,t)=&x^k+\frac{k(kn+k-1)x^{kn+k-2}}{\Gamma(\alpha+1)}t^\alpha-\frac{ax^k}{\Gamma(\beta+\alpha+1)}t^{\beta+\alpha}+\\&+\frac{\left[k(kn+k-1)(2kn+k-2)(2kn+k-3)x^{2kn+k-4}\right]}{\Gamma(2\alpha+1)}t^{2\alpha}+\\&-\frac{ak(n+2)(kn+k-1)x^{kn+k-2}}{\Gamma(2\alpha+\beta+1)}t^{2\alpha+\beta}+\\&+\frac{a^2x^k}{\Gamma(2\alpha+2\beta+1)}t^{2\alpha+2\beta}\end{aligned} \tag{8}$$

$$
\begin{aligned}
u_3(x,t) = {} & x^k + \frac{k(kn+k-1)x^{kn+k-2}}{\Gamma(\alpha+1)}t^{\alpha} - \frac{ax^k}{\Gamma(\beta+\alpha+1)}t^{\beta+\alpha} + \frac{\left[k(kn+k-1)(2kn+k-2)(2kn+k-3)x^{2kn+k-4}\right]}{\Gamma(2\alpha+1)}t^{2\alpha} + \\
& - \frac{ak(n+2)(kn+k-1)x^{kn+k-2}}{\Gamma(2\alpha+\beta+1)}t^{2\alpha+\beta} + \frac{a^2x^k}{\Gamma(2\alpha+2\beta+1)}t^{2\alpha+2\beta} + \\
& + \left[\begin{matrix} k(kn+k-1)(2kn+k-2)(2kn+k-3)\cdot \\ \cdot(3kn+k-4)(3kn+k-5) + \dfrac{1}{2}\dfrac{\Gamma(2\alpha+1)}{\left(\Gamma(\alpha+1)\right)^2} \end{matrix}\right] \frac{nk^2(kn+k-1)^2(3kn+k-4)(3kn+k-5)x^{3kn+k-6}}{\Gamma(3\alpha+1)}t^{3\alpha} + \\
& - a\left[\begin{matrix} k(n+3)(kn+k-1)(2kn+k-2)(2kn+k-3) + \\ + kn(kn+k-1)(2kn+k-2)(2kn+k-3)\dfrac{\Gamma(2\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)} \end{matrix}\right] \frac{x^{2kn+k-4}}{\Gamma(3\alpha+\beta+1)}t^{3\alpha+\beta} + \\
& + a^2\left[\begin{matrix} k(2n+3)(kn+k-1) + \\ + \dfrac{1}{2}kn(n+1)(kn+k-1)\dfrac{\Gamma(2\alpha+2\beta+1)}{\left(\Gamma(\alpha+\beta+1)\right)^2} \end{matrix}\right] \frac{x^{kn+k-2}}{\Gamma(3\alpha+2\beta+1)}t^{3\alpha+2\beta} - \frac{a^3x^k}{\Gamma(3\alpha+3\beta+1)}t^{3\alpha+3\beta}
\end{aligned} \tag{9}
$$

Finally the exact solution is obtained as:

$$u(x,t) = \lim_{n \to \infty} u_n(x,t)$$

The above series solutions generally converge very rapidly.

III. Numerical Results and Discussion

In this section, numerical results of the probability density function $u(x,t)$ for different fractional Brownian motions (FBMs) are calculated for different values of the parameters at $x = 1.1$ and these results are depicted through Figs. 1-4.

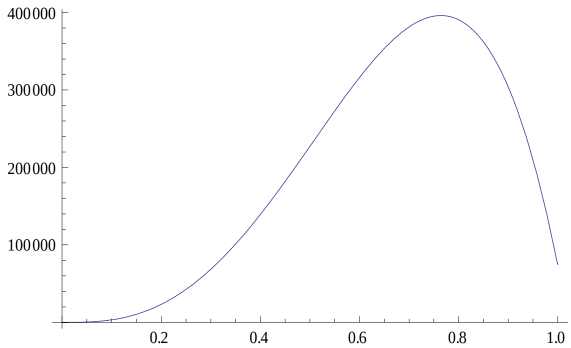

Fig. 1. Plot $u(x,t)$ vs t for $a = 540,\ \alpha = 0.95, \beta = 0.25, n = 2, k = 3,\ x = 1.1$

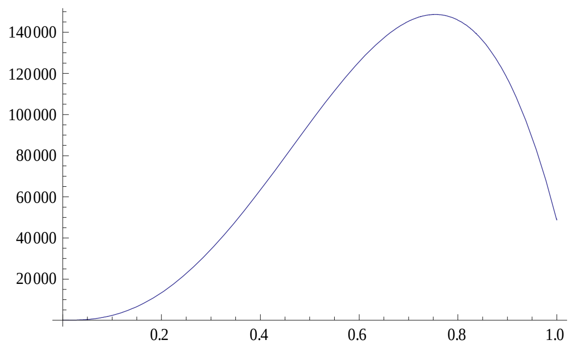

Fig. 2. Plot $u(x,t)$ vs t for $a = 395,\ \alpha = 0.95, \beta = 0.1, n = 2, k = 3, x = 1.1$

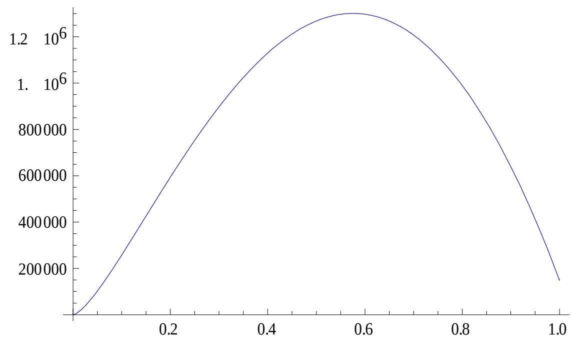

Fig. 3. Plot $u(x,t)$ vs t for $a = 450,\ \alpha = 0.5, \beta = 0.1, n = 3, k = 2.39,\ x = 1.1$

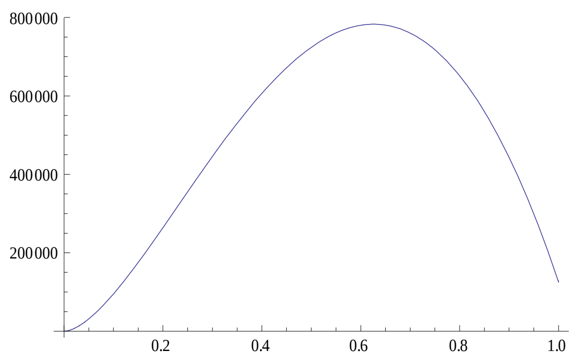

Fig. 4. Plot $u(x,t)$ vs t for $a = 392,\ \alpha = 0.6, \beta = 0.1, n = 2, k = 3.15,\ x = 1.1$

During the computation only fourth order term of variational iteration solution is used in evaluating the approximate solution of the problem. It is evident that by using more terms, the accuracies of the results can be improved.

The graphs clearly demonstrate that for various FBM ($\alpha = 0.5, 0.6, 0.95$), there are possibilities of obtaining sub-diffusion with the help of parameters like a, k, β.

It is seen from the figures that sub-diffusions occur even for the cubic order of the nonlinearity and cubic power of x in the initial condition. But beyond that, i.e., if the order of the nonlinearity and k increases, the system becomes very much unstable. It is not possible to obtain sub-diffusion even adjusting the values of other parameters.

IV. Conclusion

The article has succeeded to accomplish three important objectives. First one is the study of nonlinear fractional diffusion equation. The second one is the study of damping through numerical computation for different fractional Brownian motions due to the effect of reaction term. Third one is the successful implementation of the powerful mathematical tool VIM which provides a simple way to achieve the approximate solution of highly nonlinear fractional diffusion equation for the given complicated initial condition.

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Authors' information

Department of Applied Mathematics,
Indian Institute of Technology (BHU),
Varanasi – 221 005, India.
E-mail: subir_das08@hotmail.com