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Applications of the Variational Iteration Method to Fractional Diffusion Equations: Local versus Nonlocal Ones

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Abstract – *The diffusion equations with the local and the nonlocal fractional derivatives have been used to describe the flow through disorder media. Recently, the variational iteration method is successfully developed to find approximate solutions of the two kinds of fractional differential equations.*
This study reveals the new development of the method and compares the applications in two types of fractional diffusions.

***Keywords**: Fractional Diffusion Equation, Variational Iteration Method, Lagrange Multipliers, Caputo Derivative, Riemann-Liouville Derivative, Kolwankar-Gangal's Derivative*

Nomenclature

$u(x,t)$	Diffusion velocity, [m/s]
Abbreviations	
VIM	Variational iteration method

I. Introduction

The diffusion process has been observed in many real physical systems such as highly ramified media in porous systems, anomalous diffusion in fractals and heat transfer close to equilibrium. Fractional calculus (FC) has been a power tool to describe the diffusion behaviors in porous media.

The fractional derivative of a function depends on the values of the function over the entire interval. Thus it is suitable for describing the memory effect and modeling of the systems with long range interactions both in space and time.

On the other hand, the disorder media can be modeled by fractal method of finite scale method. In view of this point, the local fractional derivative can better describe the micro-scale behaviors which only take over the fractal media. Now there is another topic: how to develop explicit or approximate methods for solving such models. Since it's not easy to obtain the explicit solution of FDEs, numerical and approximate methods developed from differential equations and partial differential equations have been undertaking [1]-[8].

The variational iteration method developed by He in 1998 (see refs.[9],[10]) is one of the analytical methods used most often since it does need to hand nonlinear terms but still give approximate solutions of high accuracies. It has been employed in various nonlinear models arising in engineering problems [1],[11]-[19].

This study reveals the new development of the method and some diffusion equations with fractional derivatives are analytically investigated.

II. VIM I- Fractional Diffusion Equations with the Nonlocal Derivatives

According to the VIM's rules [9], [10], [20], [21], one need to follow three steps to determine the variational iteration formula: (a) establishing the correction functional; (b) identifying the Lagrange multipliers; (c) determining the initial iteration.

We can find that the crucial step is the step (b). The related properties and results of the fractional calculus can be found in [22]-[24]. We only introduce some definitions here.

Definition 2.1 [22]

The Caputo derivative is defined as:

$$ {}_0^C D_t^{\alpha} u = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{1}{(t-\tau)^{\alpha-m+1}} \frac{d^m}{d\tau^m} u(\tau)\, d\tau \qquad (1a)$$
$$0 < t, m-1 < \alpha < m, m \in Z^+$$

where Γ is the Gamma function.

Definition 2.2 [22]

The α -th Riemann-Liouville (R–L) derivative of function $u(t)$ is defined by:

$$ {}_0^{RL} D_t^{\alpha} u = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_0^t \frac{1}{(t-\tau)^{\alpha-m+1}} u(\tau)\, d\tau \qquad (1b)$$
$$m = [\alpha]+1, 0 < t, m-1 < \alpha < m, m \in Z^+$$

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Definition 2.3 [22]

The R-L integration of α order is defined by:

$$ {}_0I_t^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}u(\tau)\,d\tau \qquad (2a)$$

The Laplace transform of an original function $u(t)$ of a real variable t, is defined by the integral (if it exists):

$$\bar{u}(s)=L[u(t)]=\int_0^{\infty}e^{-st}u(t)dt,t>0 \qquad (2b)$$

where the parameter s is a complex number.

Property 2.4 [22]

The Laplace transform of the term ${}_0^CD_t^{\alpha}u$ is:

$$L\left[{}_0^CD_t^{\alpha}u\right]=s^{\alpha}\bar{u}(s)-\sum_{k=0}^{m-1}u^{(k)}\left(0^+\right)s^{\alpha-1-k}, \qquad (3)$$
$$m-1<\alpha\le m$$

II.1. Generalized Lagrange Multipliers in FC

Assume the following generalized FDEs in the Caputo's sense:

$${}_0^CD_t^{\alpha}u+R[u(t)]+N[u(t)]=k(t),0<\alpha \qquad (4)$$

where R is a linear operator, N is a nonlinear operator and $k(t)$ is a given continuous function.

Construct a correction functional:

$$u_{n+1}=u_n+\int_0^t\lambda\left({}_0^CD_\tau^{\alpha}u_n+R[u_n]+N[u_n]-k(\tau)\right)d\tau \qquad (5)$$

From Laplace transform, we can identify:

$$\lambda=\frac{(-1)^{\alpha}(\tau-t)^{\alpha-1}}{(\alpha-1)!}=-\frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!},0<\alpha \qquad (6)$$

The detail derivation can be found in [25], [26], we directly use the result here.

As a result, the iteration formula (8) is determined as:

$$\begin{aligned}u_{n+1}&=\\&=u_n+\int_0^t\frac{(-1)^{\alpha}(\tau-t)^{\alpha-1}}{(\alpha-1)!}\begin{pmatrix}{}_0^CD_\tau^{\alpha}u_n+R[u_n]+\\+N[u_n]-k(\tau)\end{pmatrix}d\tau= \quad (7)\\&=u_n-{}_0I_t^{\alpha}\left[{}_0^CD_\tau u_n+R[u_n]+N[u_n]-g(\tau)\right]\end{aligned}$$

The above formula is also valid for the FDEs with the R-L derivative.

II.2. Applications

Example 1: The Riemann-Liouville type [26]

We consider the following time-fractional diffusion equation with reaction term:

$$\begin{aligned}&{}_0^{RL}D_t^{\alpha}u(x,t)=\frac{x^2}{2}\frac{\partial^2u(x,t)}{\partial x^2}-\int_0^t\Psi(t-\tau)u(x,\tau)\,d\tau,\\&{}_0I_t^{1-\alpha}u\left(0^+\right)=x^2,0<\alpha<1\end{aligned} \qquad (8a)$$

subject to the initial condition. Here $\Psi(t)$ is the absent term and assumed as:

$$\Psi(t)=\frac{t^{\beta-1}}{\Gamma(\beta)} \qquad (8b)$$

The variational iteration formula can be given as:

$$u_{n+1}=u_n-{}_0I_t^{\alpha}\begin{pmatrix}{}_0^{RL}D_\tau^{\alpha}u_n-\frac{x^2}{2}\frac{\partial^2u_n}{\partial x^2}+\\+\int_0^t\frac{((t-\tau)^{\beta-1}}{\Gamma(\beta)}u_nd\tau\end{pmatrix} \qquad (9)$$

Set the initial value $u(x,0)=x^2$. Starting from the initial iteration $u_0(x,t)=x^2$, we can obtain the successive solutions as:

$$\begin{aligned}u_0&=x^2\\u_1&=x^2+x^2\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)\\u_2&=x^2+x^2\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)+\\&+x^2\begin{pmatrix}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}+\\+\frac{t^{2\alpha+2\beta}}{\Gamma(2\alpha+2\beta+1)}\end{pmatrix}\end{aligned} \qquad (10a)$$

$$\begin{aligned}u_3(x,t)&=x^2+x^2\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)+\\&+x^2\begin{pmatrix}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{2t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}+\\+\frac{t^{2\alpha+2\beta}}{\Gamma(2\alpha+2\beta+1)}\end{pmatrix}+\\&+x^2\begin{pmatrix}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}-3a\frac{t^{3\alpha+\beta}}{\Gamma(3\alpha+\beta+1)}+\\+3a^2\frac{t^{3\alpha+2\beta}}{\Gamma(3\alpha+2\beta+1)}-a^3\frac{t^{3\alpha+3\beta}}{\Gamma(3\alpha+3\beta+1)}\end{pmatrix}\end{aligned} \qquad (10b)$$

Example 2 The Caputo Type [25]

Consider the linear time-fractional diffusion equation:

$$ {}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t) \tag{11a} $$

with the condition:

$$ u(x,0) = sin(x),\ 0 < \alpha \le 1,\ 0<t,\ 0<x<1 \tag{11b} $$

Eq. (11a) can be used to describe the flow through porous media or a transient heat-conduction.

We have the iteration formula:

$$ u_{n+1} = u_n - {}_0 I_t^\alpha \left({}_0^C D_\tau^\alpha u_n - u_{n,xx} \right) \tag{12} $$

Starting from the initial iteration $u_0 = u(x,0) = sin(x)$, the successive approximate solutions can be obtained as:

$$ \begin{aligned} u_1 &= u_0 - {}_0 I_t^\alpha \left({}_0^C D_\tau^\alpha u_0 - u_{0,xx} \right) = \\ &= sin(x) - sin\, x \frac{t^\alpha}{\Gamma(1+\alpha)} \end{aligned} \tag{13a} $$

$$ \vdots $$

$$ u_n = sin(x) \sum_{k=0}^{n} \frac{(-t)^{k\alpha}}{\Gamma(1+k\alpha)} \tag{13b} $$

For $n \to \infty$, $u(t,x) = \lim_{n\to\infty} u_n = sin(x) E_\alpha\left(-t^\alpha\right)$ is an exact solution of Eq. (11a). Here $E_\alpha(-t^\alpha)$ is the s Mittag-Leffler function:

$$ E_\alpha\left(-t^\alpha\right) = \sum_{k=0}^{\infty} \frac{\left(-t^\alpha\right)^k}{\Gamma(1+k\alpha)} \tag{14} $$

III. VIM II-Fractional Diffusion Equations with the Local Derivatives

Fractal geometry has been used extensively in various engineering applications since it was first proposed by Madelbrot in 1973.

As is well known, methods in Euclidean geometry ordinarily deal with regular sets, while the real structures of fractal media are characterized by irregular geometry [27].

More efficient way of describing the objects is to use the Hausdorff measure of non-integer dimension.

Recently, through the Riemann-Liouville integration and Cantor-like set, Kolwankar et al.[28], [29] defined a fractional measure and proposed a local fractional derivative (LFD) and investigated the local behaviors of non-differentiable systems, especially in which the fractional order of the FC comes across the Hausdoff dimension of the Cantor-like set.

The FC can be applied to many fractal related problems in physics and engineering described by the non-where differentiable functions. Carpinteri [30] considered the diffusion problems on the Cantor-like set and derived an equation of heat diffusion as:

$$ \begin{aligned} &\frac{\partial T}{\partial t} - c\, {}^{KG}\partial_x^\alpha\, {}^{KG}\partial_x^\alpha T = 0, \\ &0 < x < 1, 0 < \alpha < 1 \end{aligned} \tag{15} $$

where c is assumed as a constant in this study and ${}^{KG}\partial_x^\alpha$ is the local fractional partial derivative. The temperature function's boundary in steady state was observed as a non-smooth case [30].

The fractional differentiable function is defined through the following measure:

$$ G(x) = {}_0 I_{C\cap[0,1]}^\alpha g(x) \chi_g(x), 0 < \alpha < 1 \tag{16} $$

where C is the Cantor-like set in the interval[0,1], ${}_0 I_x^\alpha$ is the Riemann-Liouville integration and $\chi_g(x)$ is a flag function when $x \in C$, $\chi_g(x) = 1$ and when $x \notin C$, $\chi_g(x) = 0$.

The functions generated in this way, in fact, are totally different from the analytical functions. Then the derivative is defined as:

$$ \begin{aligned} &{}^{KG}D_x^\alpha f(x) = \lim_{y\to x^+} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dy} \int_x^y \frac{f(\xi) - f(x)}{(y-\xi)^\alpha} d\xi, \\ &0 < \alpha < 1 \end{aligned} \tag{17} $$

Only when the fractional order is the Hausdorff dimension, the fractional measure and the functions' fractional derivative can exist as a non-zero and finite value. Li [31] and Chen [32] derived an equivalent form for the LFD:

$$ \begin{aligned} &{}^{KG}D_x^\alpha f(x) = \lim_{y\to x^+} \frac{\Gamma(1+\alpha)\left(f(y) - f(x)\right)}{(y-x)^\alpha} \\ &0 < \alpha < 1 \end{aligned} \tag{18} $$

We use the definition (18) for simplicity in this paper. Note that Adda [33] and Jumarie [34],[35] also used and derived this definition for non-differentiable functions.

One can derive the Leibniz law, the integration by parts for the fractional differentiable functions et al. However, the existing results mainly concentrate on the fractional differentiable functions' LFD at the initial point $x = 0$. In fact, a binomial expansion [36] is needed to extend the initial point to any points on the fractal set in the interval

[0,1] and then we can calculus the fractional differentiable functions' LFD at any points:

$$ {}^{KG}D_x^{\alpha}G(x)=g(x)\chi_g(x) \tag{19}$$

On the other hand, one can check its correctness using the mean-value theorem [32],[37].

Now let's apply the VIM. Establish a correction functional for Eq. (15):

$$T_{n+1}=T_n+{}_0I_x^{\alpha}\lambda(\xi,x)\left[\frac{\partial T_n}{\partial t}-c\,{}^{KG}\partial_{\xi}^{\alpha}\,{}^{KG}\partial_{\xi}^{\alpha}T_n\right] \tag{20a}$$

where $\lambda(\xi,x)$ is to be determined. Consider the term $\frac{\partial T_n}{\partial t}$ as a restrict variation, via the variation of calculus, one can derive the equations with respect to $\lambda(\xi,x)$:

$$1-\lambda^{(\alpha)}|_{\xi=x}=0,\ \lambda^{(2\alpha)}=0\ \text{ and }\ \lambda|_{\xi=x}=0 \tag{20b}$$

Now the generalized Lagrange multiplier can be identified as:

$$\lambda(\xi,x)=\frac{(\xi-x)^{\alpha}}{\Gamma(1+\alpha)} \tag{21}$$

Assume $T(0,t)=c_0(t)$ and ${}^{KG}\partial_{x=0}^{\alpha}T(x,t)=c_1(t)$, then the initial iteration can be derived via the Taylor series [38]:

$$T_0=c_0(t)+c_1(t)\frac{x^{\alpha}}{\Gamma(1+\alpha)} \tag{22}$$

Now, the iteration formula for (20a) now can be completely given as:

$$T_{n+1}=T_n+{}_0I_x^{\alpha}\frac{(\xi-x)^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{\partial T_n}{\partial t}-c\,{}^{KG}\partial_{\xi}^{\alpha}\,{}^{KG}\partial_{\xi}^{\alpha}T_n\right] \tag{23}$$

The successive approximation of non-analytical can be given:

$$\begin{aligned} T_0&=c_0(t)+c_1(t)\frac{x^{\alpha}}{\Gamma(1+\alpha)}\\ T_1&=c_0(t)+c_1(t)\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\\ &+c_0'(t)\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+c_1'(t)\frac{x^{3\alpha}}{\Gamma(1+3\alpha)} \end{aligned} \tag{24}$$

In the above derivation, the integration by parts [34] for the fractional differentiable functions is used.

For example, in the derivation of T_1, we can find that:

$$\begin{aligned} T_1&=T_0+{}_0I_x^{\alpha}\frac{(\xi-x)^{\alpha}}{\Gamma(1+\alpha)}\left[c\,{}^{KG}\partial_{\xi}^{\alpha}\,KG\partial_{\xi}^{\alpha}T_0-\frac{\partial T_n}{\partial t}\right]=\\ &=c_0(t)+c_1(t)\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\\ &+{}_0I_x^{\alpha}\frac{(\xi-x)^{\alpha}}{\Gamma(1+\alpha)}\left[c_0'(t)+c_1'(t)\frac{\xi^{\alpha}}{\Gamma(1+\alpha)}\right] \end{aligned} \tag{25a}$$

where:

$$ {}_0I_x^{\alpha}\frac{(\xi-x)^{\alpha}}{\Gamma(1+\alpha)} \tag{25b}$$

and:

$$ {}_0I_x^{\alpha}\left[\frac{(\xi-x)^{\alpha}}{\Gamma(1+\alpha)}\frac{\xi^{\alpha}}{\Gamma(1+\alpha)}\right] \tag{25c}$$

can be calculated as:

$$ {}_0I_x^{\alpha}\frac{(\xi-x)^{\alpha}}{\Gamma(1+\alpha)}=\left.\frac{(\xi-x)^{2\alpha}}{\Gamma(1+2\alpha)}\right|_0^x=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \tag{25d}$$

$$\begin{aligned} &{}_0I_x^{\alpha}\left[\frac{(\xi-x)^{\alpha}}{\Gamma(1+\alpha)}\frac{\xi^{\alpha}}{\Gamma(1+\alpha)}\right]=\\ &=\left[\left.\frac{\xi^{2\alpha}}{\Gamma(1+2\alpha)}\frac{(\xi-x)^{\alpha}}{\Gamma(1+\alpha)}\right|_0^x-{}_0I_x^{\alpha}\left.\frac{\xi^{2\alpha}}{\Gamma(1+2\alpha)}\right|_0^x\right]=\\ &=-\frac{x^{3\alpha}}{\Gamma(1+3\alpha)} \end{aligned} \tag{25e}$$

As a result, the 1st term approximation can be obtained:

$$\begin{aligned} T_1&=c_0(t)+c_1(t)\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\\ &+c_0'(t)\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}-c_1'(t)\frac{x^{3\alpha}}{\Gamma(1+3\alpha)} \end{aligned} \tag{26}$$

Substituting T_1 into Eq. (23), then T_2 can be determined.

The higher term iteration solutions can be derived without any difficulty. Readers must note that the functions $x^{\alpha}, x^{2\alpha}, x^{3\alpha}$ above are fractional differentiable functions. We only described the figures of the approximate solutions on the large-scale [39]-[41], i.e., the $L'_0(x)$ in Fig. 1.

Kolwankar plotted the shape of x^{α} through the Lebesgue decomposition method [42].

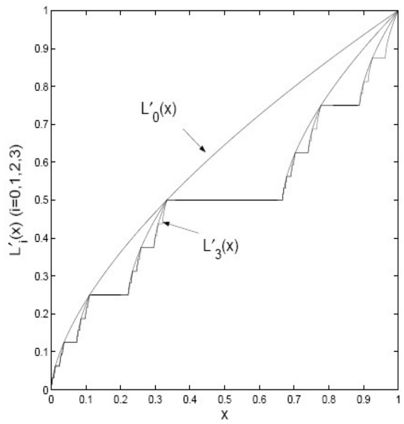

Fig. 1. The shape of the fractional differentiable function x^{α} [42]

$L'_0(x)$ is plotted when the fractal scale 1 and $L'_3(x)$ is plotted when the fractal scale is 1/27. Thus, the graph functions of $x^{2\alpha}, x^{3\alpha}, ..., x^{k\alpha}$ are a class functions of the Devil's staircase in the non-differentiable approximation.

IV. Conclusion

The variational iteration method has been used by many researchers and has matured into a powerful tool to deal with nonlinear equations. But the method wasn't successfully used like other popular analytical methods in fractional calculus due to incorrect identifications of the Lagrange multipliers or the weighted functions.

This study mainly concentrates on the applications to fractional differential equations: the Caputo, the R-L and the Kolwankar-Gangal's derivatives which are nonlocal and local fractional derivatives, respectively. In order to apply the method, reads need to know the differences between each other:

(a) The physical meanings of the fractional derivatives are different. For the local fractional derivative, the Kolwankar-Gangal's derivative, the fractional order is the fractal dimension. So the generated functions through the R-L integration and the Cantor-like set are non-differentiable or fractional analytical functions.

(b) The physical meanings of approximate solutions are different. For the nonlocal one, the approximate solutions are "smooth". They are not suitable for fractal boundary value problems. The Caputo derivative only deals with smooth initial boundary problems. For the Kolwankar-Gangal's derivative, the approximate solutions are "non-smooth" so that they can describe micro and self-similar behaviors i.e., the crack curves, flows in porous medium on a nano-scale et al. As a result, the VIMII indeed is a multi-scale method due to the changing scales of the infinite fractal media.

However, there still a long way for one to use the VIMII in fractal boundaries of engineering problems.

The time and the space should be dimensionless. The approximate solutions need be described by graph functions using the Lebesgue decomposition method [42] on different fractal scales while the approximate solutions in the nonlocal case are analytical and they can be employed and plotted directly.

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