

An Exercise with the He's Variation Iteration Method to a Fractional Bernoulli Equation Arising in Transient Conduction with Non-Linear Heat Flux at the Boundary

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Abstract – Surface temperature evolution of a body subjected to a nonlinear heat flux involving counteracting convection heating and radiation cooling has been solved by the variations iteration method (VIM) of He. The surface temperature equations comes as a combination of the time-fractional (half-time) subdiffusion model of the heat conduction and the boundary condition relating the temperature field gradient at the surface through the Riemann-Liouville fractional integral. The result of this equation is a Bernoulli-type ordinary fractional equation with a non-linear term of 4th order. Two approaches in the identification of the general Lagrange multiplier and a consequent application of VIM have been applied. The common method replaces the fractional integral (in the Riemann-Liouville sense) by an integer order integral. The more exact method of Wu uses an initial Laplace transform of the fractional integral in the iteration formula to find correctly the Lagrange multiplier. Both approaches yield different Lagrange multipliers which results in different numerical results. The article discusses the origin of these differences in the light of correct application of VIM. The developed approximate solutions have been used in numerical simulation showing the effect of Biot number and the radiation-conduction number on the surface temperature evolution. **Copyright © 2012 Praise Worthy Prize S.r.l. - All rights reserved.**

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Nomenclature

a	Thermal diffusivity, m ² /s
b	Sample (slab) thickness, m
h	Heat transfer coefficient, W/m ² K
k	Thermal conductivity, W/mK
$q_s^n(t)$	Surface flux density, W/m ²
t	Time, s
$t^* = b^2/a$	Time scale (heat diffusion), s
T_a	Ambient temperature, K
T_{ref}	Reference temperature, K
T_s	Surface temperature, K
T_∞	Initial temperature of the body, K
x	Space co- ordinate, m
$y(t)$	Casual function
Dimensionless Numbers	
$Bi = bh/k$	Biot number
$ Fo = t/t^* = ta/b^2$	Fourier number
$ N = b\sigma\epsilon T_a^3/k$	Radiation conduction number

Greek Letters

α	Fractional order, dimensionless
ϵ	Emissivity of cooled surface, dimensionless
$\lambda(\tau)$	General Lagrange multiplier
λ_H	Lagrange multiplier defined by the method of He (see also the articles of S. Momani et al.)
λ_W	Lagrange multiplier defined by the method of G. C. Wu [24], [25] (see Eq.(13c))
σ	Stefan Boltzmann constant
$\Gamma(\bullet)$	Gamma function
$\theta = T_s/T_a$	Dimensionless temperature
$\bar{\theta}(s)$	Laplace transform of the dimensionless temperature θ
$\xi = x/b$	Dimensionless space co-ordinate
τ	Dummy variable in the fractional integrals and derivatives

Subscripts

n	Number of iteration
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Superscripts

<i>C</i>	Cooling
<i>H</i>	Heating
<i>R</i>	Radiative

I. Introduction

The heat transfer problems are among those with intensive applications of new analytical and numerical methods for solving new emerging practical situations.

The present article addresses two principle issues. First of all, a problem of surface heating of body subjected to a non-linear heat flux compromising a convective heating and a counteracting radiation cooling is formulated. The problem formulation uses half-time fractional model (in the Riemann-Liouville sense) to create an ordinary fractional differential equation (OFDE).

This allows elimination of cumbersome solutions of the temperature file in the depth of the thermally treated body. Second, the fractional model of surface temperature evolution is of Bernoulli type OFDE which is solved by the variational iteration method (VIM) of He. The main problem is the correct identification of the general Lagrange multiplier. All these problems are consequently developed in the article.

1.1. Non-Linear Boundary Condition due to Convection and Thermal Radiation

The transient heating of flat surfaces is a common thermal problem solved by many methods and under various boundary conditions.

It is impossible to encompass all of them, relevant especially to the flash heating of bodies used in the chemical, metallurgical and mineral industries. We stress the attention to 1-D transient heat conduction with a nonlinear boundary condition in the form:

$$-k \left. \frac{\partial T_s}{\partial x} \right|_{x=0} = h(T_s - T_{ref}^C) \pm \sigma \varepsilon (T_s^4 - T_{ref}^R) \quad (1a)$$

In (1a) the reference temperature is [1],[2] is defined in accordance with the direction of heat fluxes towards/from the interface, namely:

for cooling:

$$T_{ref}^C = T_a \text{ and } T_{ref}^R = T_a \quad (1b)$$

for heating:

$$T_{ref}^H = T_a \text{ and } T_{ref}^R = T_R \quad (1c)$$

In the light of application of the He's VIM (see refs [3]-[8]) to solve nonlinear conduction equations this boundary condition has been recently used for solving convective-radiative cooling of lumped system (fin conduction problem) [9],[10]; as an independent

equation with zero initial condition. Now, we stress the attention on a fractional-time equation arising in transient heating of bodies where (1a) is related to the time-evolution of surface temperature and the solution is developed without the temperature field in the depth of the body.

II. Problem Formulation and Model

The flash heating of powder for example is a technique widely applied for de-hydroxylation of kaolinite. This mineral ($\text{Al}_2\text{O}_3 \cdot 2\text{SiO}_2 \cdot 2\text{H}_2\text{O}$) undergoes an endothermic reduction (loss) of the amount of the hydroxyl water at about 600°C resulting in meta-kaolin.

This final form is strongly dependent on the rate of heating during the flash process. The kaolinite particles are in the micro-size range but form aggregates that get forms of flakes which to some extent can be accepted as plates when the thermal treatment process has to be modelled. In accordance with the model of Davies [11] the kaolinite aggregates are assumed as plates of thickness b heated by convection passing rapidly through a furnace and cooled by thermal radiation. It is well-known that such models have no analytical solutions due to the non-linearity of the boundary conditions at the interface.

The process of flash heating of plate relevant to thermal treatment of kaolinite has been modelled by Davies [11] and solved approximately by the heat-balance integral method of Goodman [12].

The same method has been applied to radiation cooling of finite solids [13] and plates by convection-radiation [14]. The heat balance integral solution allows the surface temperature to be estimated by developing an approximated temperature profile in the depth of body, then defining the temperature gradient required to solve (1a) with respect to T_s . The main drawback of this solution is that the temperature gradient at the interface depends on the choice of the approximate profile [12] and its calibration [15],[16].

The present work considers the transient heat conduction in a plate of thickness b which is heated by convection and cooled by thermal radiation. This is, in fact, the problem considered by Davies [11], namely

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2}, \quad a = \frac{k}{\rho C_p}, \quad 0 \leq x \leq b \quad (2a)$$

with:

$$\frac{\partial T}{\partial x} = 0 \text{ at } x = b \quad (2b)$$

and:

$$-k \left. \frac{\partial T_s}{\partial x} \right|_{x=0} = h(T_s - T_a) - \sigma \varepsilon (T_s^4 - T_a^4) \text{ at } x = 0 \quad (2c)$$

The problem of interest is the time evolution of the surface temperature T_s which to some extent could be accepted with an initial condition $T_s(t=0)=0$ for simplicity of the solution. It can be solved by either approximate analytical methods as those mentioned above or numerically. However, with all these methods the main errors come from the approximate determination of the space derivative $(\partial T/\partial x)_{x=0}$ [12], [15],[16]. In order to find an effective solution of the problem where only the surface temperature T_s is of interest (it is the upper limit to which the body should be heated; determined by technological conditions, for example) we refer to the possibility to split (2a) into a fractional (half-time) heat conduction equation, namely:

$$\frac{\frac{1}{\partial^2 T}(x,t)}{\frac{1}{\partial t^2}} = -\sqrt{a} \frac{\partial T(x,t)}{\partial x} \quad (3a)$$

with a fractional half-time derivative in the Riemann-Liouville sense:

$${}^{RL}D_t^{1/2} = \frac{\partial^{1/2} T(x,t)}{\partial_+ t^{1/2}} = \frac{1}{\Gamma(1/2)} \frac{d}{dt} \int_0^t \frac{T(x,u)}{\sqrt{t-u}} du \quad (3b)$$

The surface flux and the temperature at the interface are interrelated by (3a) setting $x=0$:

$$q_s''(0,t) = \frac{k}{\sqrt{a}} \left[\frac{\partial^{1/2} T(0,t)}{\partial t^{1/2}} - \frac{T_\infty}{\sqrt{\pi t}} \right] \quad (4a)$$

with $T_{(x=0)} = T_s$

$$T(0,t) = \frac{\sqrt{a}}{k} \frac{\partial^{-1/2} [q_s''(0,t)]}{\partial t^{-1/2}} + T_\infty \quad (4b)$$

with:

$$-k \frac{\partial T}{\partial x} \Big|_{x=0} = q_s''(t)$$

T_∞ in (4a),(4b) is the initial temperature of the body. Therefore, expressing the surface flux by (4a) and the boundary condition (2c) we get:

$$-\frac{k}{\sqrt{a}} \left[\frac{\partial^{1/2} T_s}{\partial t^{1/2}} - \frac{T_\infty}{\sqrt{\pi t}} \right] = h(T_s - T_a) - \sigma \varepsilon (T_s^4 - T_a^4) \quad (5a)$$

at $x=0$.

In accordance with the assumption that $T_s(t=0)=0$ we may suggest $T_\infty=0$ in (4a),(4b) that simplifies the

problem at issue.

► **Note:** In order to be correct, a variable $u = T_\infty - T$ transforms the initial condition of Eq. (2a) to zero one, which simplifies the problem with application of the RL derivative. Similar problem, with only radiative cooling, i.e., $q_s = (-\partial T/\partial x)_{\partial x=0} = \mu T_s^4$ and the variable $u = T_\infty - T$ through the relationship (4b) yields $T_s(\tau^*) = T_\infty - \mu D^{-1/2} T_s^4(\tau^*)$, where τ^* is dimensionless time. The variable $u = T_\infty - T$ means only a shift of the origin of the co-ordinate systems, that is equivalent to the assumption that $T_\infty = 0$ at $t=0$:

$$-\frac{k}{\sqrt{a}} \frac{\partial^{1/2} T_s}{\partial t^{1/2}} = h(T_s - T_a) - \sigma \varepsilon (T_s^4 - T_a^4) \quad (5b)$$

Equation (5a) is expressed in a dimensional form but after non-dimensionalization of the basic model (2a,b,c) with $\theta = T_s/T_a$, $\xi = x/b$ and $Fo = t/t^*$ ($t^* = b^2/a$), we get:

$$\frac{\partial \theta}{\partial Fo} = \frac{\partial^2 \theta}{\partial \xi^2}, \quad 0 \leq \xi \leq 1 \quad (6a)$$

with:

$$\frac{\partial \theta}{\partial \xi} = 0 \text{ at } \xi = x/b = 1 \quad (6b)$$

and:

$$-\frac{\partial \theta_s}{\partial \xi} \Big|_{x=0} = Bi(\theta_s - 1) - N(\theta_s^4 - 1) \text{ at } x=0 \quad (6c)$$

$$Bi = \frac{bh}{k} \quad - \text{ Biot Number} \quad (7a)$$

$$N = \frac{b\sigma\varepsilon T_a^3}{k} \quad - \text{ Radiation-conduction number} \quad (7b)$$

Now, using the relationship (4a) in a dimensionless form we get:

$${}^{RL}D_{Fo}^{1/2} \theta_s = Bi(\theta_s - 1) - N(\theta_s^4 - 1), \quad \theta_{s(Fo=0)} = 0 \quad (8a)$$

After rearranging the RHS of (8a) we get a fractional (half-time) Bernoulli equation, namely:

$${}^{RL}D_{Fo}^{1/2} \theta_s = A + B\theta_s + C\theta_s^4 \quad (8b)$$

$$A = N - Bi, \quad B = Bi \text{ and } C = -N \quad (8c)$$

This fractional-time Bernoulli equation was not encountered in the literature so far, even with arbitrary fractional order $0 < \alpha < 1$ similar to the fractional Riccati equation; expressed through the Caputo derivatives [16] or modified Riemann-Liouville (Jumarie's) derivatives [17], namely:

$${}_0^C D_t^\alpha y(t) = A_1(t) + B_1(t)y + C_1(t)y^2, \quad 0 < \alpha < 1 \quad (9)$$

III. Solution by VIM

In accordance with the VIM methodology, we have to construct a correction functional for (8b) in the form:

$$\theta_{n+1} = \theta_n + {}_0^{RL} I_t^\alpha \left[\lambda(\tau) \left({}_0^{RL} D_{Fo}^\alpha \theta_n - A + \right) \right] \quad (10a)$$

where:

$${}_0^{RL} I_t^\alpha \theta(Fo) = \frac{1}{\Gamma(\alpha)} \int_0^{Fo} (Fo - \tau)^{\alpha-1} \theta(Fo) d\tau \quad (10b)$$

is a fractional integral in the Riemann-Liouville sense.

The crucial step defining the successful application of VIM is the determination of the general Lagrange multiplier $\lambda(\tau)$. The problems existing in its determination are discussed next.

III.1. Lagrange Multiplier Determination

The main problem emerging through the identification of the general Lagrange multiplier in case of fractional equations is the fact that with the integral I^α in the correctional functional the integration by parts cannot be applied [3],[4]. We will discuss two approaches existing in the literature and providing different Lagrange multipliers.

Approach 1:

Following Momani and Odibat [19]-[21] and other articles devoted to similar problems [17], [18], [22], [23] the iteration formula (10a) is replaced by:

$$\theta_{n+1} = \theta_n + \int_0^{Fo} \lambda(\tau) \left[{}_0^{RL} D_{Fo}^\alpha \theta_n - A - B\tilde{\theta}_n - C\tilde{\theta}_n^4 \right] d\tau \quad (11a)$$

The stationary condition of the functional leads too:

$$\delta\theta_{n+1} = \delta\theta_n + \delta \int_0^{Fo} \lambda(\tau) \frac{d\theta_n}{d\tau} d\tau = 0 \quad (11b)$$

Because $\tilde{\theta}_n$ is a restricted variation, i.e. $\delta\tilde{\theta}_n = 0$, the condition (11b) yields: $\lambda'(\tau) = 0$ and $1 + \lambda(\tau) = 0$

resulting in $\lambda = -1$. The result is the same as for the fractional Riccati equations [17],[18] because both the linear and the non-linear terms in the RHS of (11b) are restricted variations. Therefore, the correctional functional (10a) for $0 < \alpha < 1$ (omitting the subscript Fo) should be:

$$\theta_{n+1} = \theta_n - {}_0^{RL} I_t^\alpha \left[{}_0^{RL} D^\alpha \theta_n - A - B\theta_n - C\theta_n^4 \right] \quad (11c)$$

However, following the rules defined by (11a),(11b) we may write the correction functional also as:

$$\theta_{n+1} = \theta_n - \int_0^{Fo} \left[{}_0^{RL} D_{Fo}^\alpha \theta_n - A - B\tilde{\theta}_n - C\tilde{\theta}_n^4 \right] d\tau \quad (11d)$$

i.e. the Riemann-Liouville integral is replaced by an integer-order Riemann integral

Therefore, there are two options in the successive iterations after determination of the Lagrange multiplier, namely:

- 1) **Approach 1 - Option 1 (A1-O1)**, $\lambda_H = -1$ and Riemann integration by (11d)
- 2) **Approach 1 - Option 2 (A1-O2)**, $\lambda_H = -1$ and Riemann-Liouville integration by (11c)
 - The subscript "H" indicate that this multiplier is identified by rules drawn by J.H-He (see III.3 for detailed comments)

These two options will be commented and numerically exemplified further in this article.

Approach 2:

Avoiding the problem with the impossible integration by parts with the integral I^α in (10a), we apply the approach of Wu [24], [25] which has two principle steps:

- a) By an initial Laplace transform to (10a) we get:

$$\bar{\theta}_{n+1}(s) = \bar{\theta}_n(s) + L \left[\int_0^t \lambda(t, \tau) \left({}_0^{RL} D^\alpha \theta_n - A + \right) \right] \quad (12a)$$

The Lagrange multiplier is assumed in the form $\lambda(t, \tau) = \lambda(t - \tau)$ allowing:

$$L \left[\int_0^t \lambda(t, \tau) \left({}_0^{RL} D^\alpha \theta_n - A - B\theta_n - C\theta_n^4 \right) \right] \quad (12b)$$

to be considered as convolution of $\lambda(t)$ and:

$$\left({}_0^{RL} D^\alpha \theta_n - A - B\theta_n - C\theta_n^4 \right)$$

Then, if $\tilde{\theta}_n$ and $\tilde{\theta}_n^4$ are restricted variations (similar to Approach 1), we get:

$$\begin{aligned} \delta\bar{\theta}_{n+1}(s) &= \delta\bar{\theta}_n(s) + \delta L \left[{}_0 I_t^\alpha \lambda \left({}^R L_0 D^\alpha \theta_n \right) \right] = \\ &= \left(1 + \bar{\lambda}(s) s^\alpha \right) \delta\bar{\theta}_n(s) \end{aligned} \quad (12c)$$

The stationary conditions implies the coefficient of $\delta\bar{\theta}_n(s)$ to be equal to zero, that results in:

$$\bar{\lambda}(s) = -\frac{1}{s^\alpha} \quad (13a)$$

b) The inverse Laplace transform of (13a) gives the Wu's multiplier [24], [25], (denoted as λ_W):

$$\lambda_W(t, \tau) = -\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} = \frac{(-1)^\alpha (t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \quad (13b)$$

More detailed analyses about this approach and the Lagrange multiplier (13b) are available in [24], [25].

Thus, with the Lagrange multiplier (13b) and taking into account that $t \equiv Fo$ (and omitting the subscript Fo), the correctional functional (10a), for $0 < \alpha < 1$, is:

$$\begin{aligned} \theta_{n+1} &= \theta_n + \\ &+ \int_0^{Fo} \left[\frac{(-1)^\alpha (Fo-\tau)^{\alpha-1}}{\Gamma(\alpha)} \right] \times \left[{}^R L_0 D^\alpha \theta_n - A + \right. \\ &\left. - B\theta_n - C\theta_n^4 \right] d\tau \end{aligned} \quad (13c)$$

III.2 Iterations

Approach 1 (with $\lambda_H = -1$)

Option 1 (A1-O1, $\lambda_H = -1$ and Riemann integration by (11d).

This is common approach well-known from the literature (see refs.[17]-[22]). Assuming the initial approximation as $\theta_0 = \frac{Fo^\alpha}{\Gamma(1+\alpha)}$ we get by iteration:

$$\begin{aligned} \theta_1^{A_1-O_1} &= (A-1)Fo + \frac{Fo^\alpha}{\Gamma(1+\alpha)} + \\ &+ B \frac{Fo^{\alpha+1}}{(1+\alpha)\Gamma(1+\alpha)} + C \frac{Fo^{5\alpha+1}}{(1+5\alpha)[\Gamma(1+\alpha)]^4} \end{aligned} \quad (14a)$$

with $\alpha = 1/2$, we get:

$$\begin{aligned} \theta_1^{A_1-O_1} &= (A-1)Fo + \frac{Fo^{1/2}}{\Gamma(3/2)} + \\ &+ B \left(\frac{2}{3} \right) \frac{Fo^{3/2}}{\Gamma(3/2)} + C \left(\frac{2}{7} \right) \frac{Fo^{7/2}}{[\Gamma(3/2)]^4} \end{aligned} \quad (14b)$$

3) Option 2 (A1-O2), $\lambda_H = -1$ and Riemann-Liouville

integration by (11c), yield.

Moreover, taking into account (see (8c)) that the coefficients in the Bernoulli equations are $A = N - Bi$, $B = Bi$ and $C = -N$, and with $\alpha = 1/2$ we get:

$$\begin{aligned} \theta^{A_1-O_2} &= (N - Bi) \frac{Fo^{1/2}}{\Gamma(3/2)} + Bi \frac{Fo}{\Gamma(2)} + \\ &- N \frac{Fo^{5/2}}{\Gamma(7/2)} \frac{\Gamma(3)}{[\Gamma(3/2)]^4} \end{aligned} \quad (15)$$

The experiment just performed, was done by intuition, i.e. a mechanistic combination of $\lambda_H = -1$ with the Riemann-Liouville integration. However, as it is demonstrated further in this article (see sec. V) this is a route equivalent to a combination of the Wu's multiplier and the Riemann integration.

Approach 2 (with $\lambda_W = -\frac{(Fo-\tau)^{\alpha-1}}{\Gamma(\alpha)}$) and integration by (13d)

With the same initial approximation $\theta_0 = \frac{Fo^\alpha}{\Gamma(1+\alpha)}$

and taking into account that $t \equiv Fo$ we get by iteration

$$\begin{aligned} \theta_1^W &= A \frac{Fo^\alpha}{\Gamma(1+\alpha)} + B \frac{Fo^{2\alpha}}{\Gamma(1+2\alpha)} + \\ &+ C \frac{Fo^{5\alpha}}{\Gamma(1+5\alpha)} \frac{\Gamma(1+4\alpha)}{[\Gamma(1+\alpha)]^4} \end{aligned} \quad (16a)$$

with $\alpha = 1/2$ we have:

$$\theta_1^W = A \frac{Fo^{1/2}}{\Gamma(3/2)} + B \frac{Fo}{\Gamma(2)} + C \frac{Fo^{5/2}}{\Gamma(7/2)} \frac{\Gamma(3)}{[\Gamma(3/2)]^4} \quad (16b)$$

The these results, i.e. (14b), (15) and (16b) are presented graphically in a series of plots (see Figs. 1-3) with various values of Bi and N numbers.

III.3 Comments on the Lagrange Multipliers

The different results developed by both approaches need clarifications about the origin of these results. First of all it is clear that with the fractional integral and a fractional derivative in the functional the differentiation by parts does not hold. Browsing the literature on VIM, several principle approximations have been done, among them:

The first approximation in the identification of the general (example 2 in ref.[26]) provoked a simplification by introducing restricted variations of high-order terms in the correction functional. This step, which, in fact,

simplified the integration by parts, yields an approximate Lagrange multiplier. Now, this is a common step in all studies on VIM, but commonly the authors forget the reasons for this approach.

The second approximation in application of VIM was done when fractional differential equations were at issue (see in detail part 5 in ref. [27]). Since the integration by parts does not hold with the Riemann-Liouville integral and the fractional derivative in the functional, He did two simplifications more a) Replaced the Riemann-Liouville integral by an integer Riemann Integral; b) The fractional derivative was replaced by an integer one of order m (minimal $m = \text{ceil}(\alpha) > \alpha$ or maximal integer $m = \text{floor}(\alpha) < \alpha$). In the case of $0 < \alpha < 1$, and particularly with $\alpha = 1/2$ we have $m = \text{floor}(\alpha) = 1$.

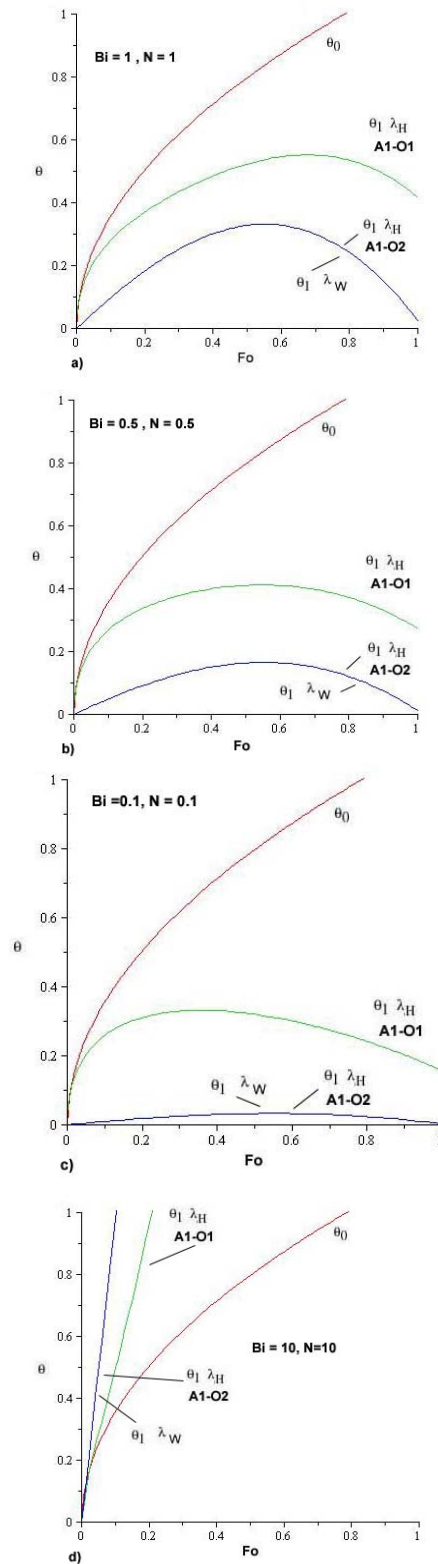
Combining Approx. 1 and Approx. 2, and the ideas of He [26], [27] the steps expressed by Eqs (11a), (11b) seems correct even though these are approximations avoiding principle obstacles in determination of the Lagrange multiplier. However, the consequent application of an integer (Riemann) integral in the development of the successive iteration (see again Chapter 5 in Ref. [2]) does not mean reasonable from a mathematical point of view.

Nevertheless, from practical reasons, there is strong motivation this to be done. Simply, the increase in the order of iterations makes the calculations cumbersome which are hard to be handled manually, but easily carried out by *Maple* or *Mathematica*, where fractional integrals are not developed. Frequently, in the recent literature, this approach is referred to the works of Momani and collaborators [19]-[21] since they are easily accessible, but, in fact, the origin is the works of He [26], [27].

The Wu's approach attacks the problem correctly, since it strictly follows the VIM's rules without any approximation in the type of the integral in the correction functional. The only trace remaining from the He's suggestions is the Approximation 1, with restricted variations of high order terms.

IV. Numerical Simulations and Comments

Before performing cumbersome iterations resulting from the non-linear radiation term in the boundary condition, let us see what is the effect of Bi and N dimensionless numbers on the behaviour of the first iteration performed with the two distinct Lagrange multipliers and the suggested option in the integration in the correction functional. Initially, we start with $Bi = 1$ and $N = 1$, following the work of Davies [11] and Crosbie and Viskanta [28]. The plots in Fig. 1(a) indicate that the first iteration with $\lambda_H = -1$ goes far away from the initial approximation, while with λ_W and $\lambda_H (A_1 - O_2)$ the lines are practically indistinguishable.

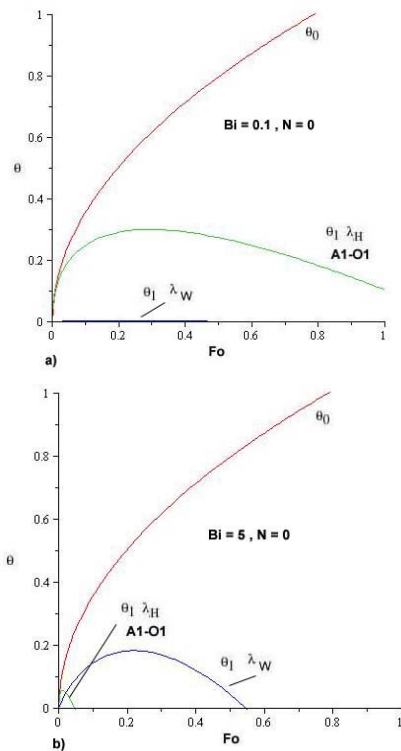


Figs. 1. Effect of the Lagrange multiplier on the first iteration in various case of convective heating and radiative cooling

The same behaviour is observed with $N = 0.5$ (see Fig. 1(b)), as well as when $Bi = 0.1$ and $N = 0.1$ (Fig. 1(c)). The case with extreme values of $Bi = 10$ and ($N = 10$) does give adequate information which of the

first iterations is better (Fig. 1(d)).

In case of only heating ($N = 0$) (see Figs. 2(a), (b)); it is possible to distinguish the behaviour of the lines, but in all these the first iterations with $\lambda_H (A_1-O_2)$ and the Wu's, multiplier and consequent Riemann-Liouville integration practically merge (indistinguishable).



Figs. 2. Effect of the Lagrange multiplier on the first iteration in various case of convective heating only

The cases with small Bi numbers are small and the radiative cooling is dominating (see Fig. 3(a)) demonstrate distinguishable behaviour of the first iterations.

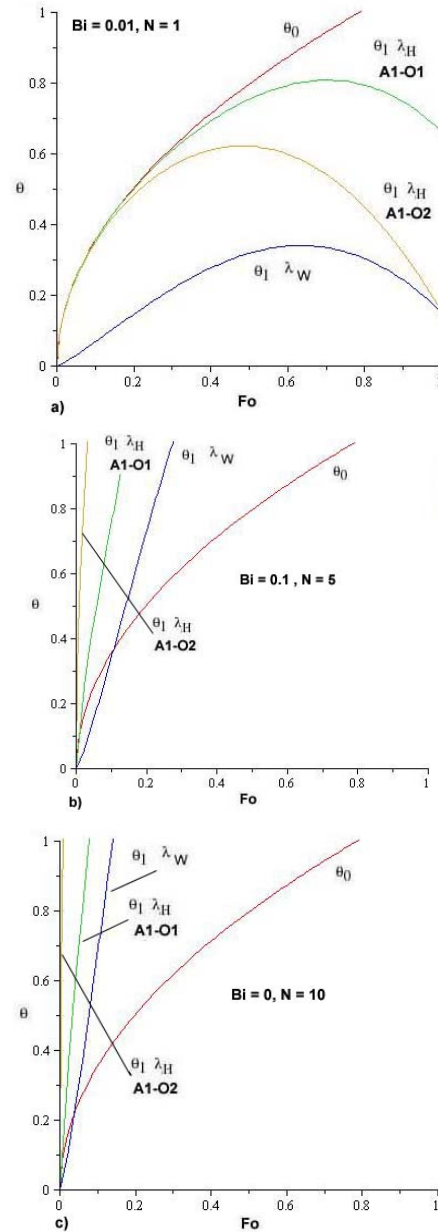
In the other cases (Figs. 3(b), (c)) it is hard only on the basis of the first iteration to get right information which approximation is better, even though theoretically that of Wu with λ_W is the correct one.

These are only first attempts to find what should be done further.

The large varieties of values of the dimensionless numbers Bi and N do not allow to estimate, at this moment, the best performance of the approximation with either λ_H or λ_W . However, this can be simply estimated in a particular case strongly related to a practical problem.

A special attempt in this work was done by non-dimensionalization of the governing equations and using the Fourier number Fo as independent variable. This was done, because, it is well-known that convergence of VIM is better at small times, i.e. at small values of Fo in the present case.

Physically, small Fo , means either small times for materials with small thermal diffusivity time scale (b^2/a), or large times with large b^2/a . In any case, the use of Fo as independent variable allows VIM to be applied.



Figs. 3. Effect of the Lagrange multiplier on the first iteration in various case of radiative cooling only

The problems with successive approximation of high order and their convergences when both Lagrange multipliers are applied is open and its solution is a challenge, but beyond the scope of the present work. The better way, to elucidate this crucial for the VIM's applications problem is to solve well-known examples available in the literature; this draws ideas for future studies.

V. Some Comments on the Options in the Integration and the Emerging Results

The previous point demonstrated what the numerical behaviour of the first iteration is when some combinations (option in iterations) between the Lagrange multiplier and the type of the integral were performed.

First of all, starting with the dominating approach used by Momani et al. ([19]-[21]), we have:

$$u_{n+1} = u_n + \int_0^t \lambda(t, \tau) \left[\begin{array}{l} {}_0^C D_\tau^\alpha u_n + R[\tilde{u}_n] + \\ + N[\tilde{u}_n] - f(\tau) \end{array} \right] d\tau \quad (17a)$$

For the equation:

$${}_0^C D_\tau^\alpha u_n + R[\tilde{u}_n] + N[\tilde{u}_n] = f(\tau) \quad (17b)$$

Here we use the common notations available in the literature.

With the Lagrange multiplier $\lambda_H = -1$ commonly used in the literature (Momani [19]-[21]), we get results already demonstrated by integration with the Riemann integral, namely:

$$u_{n+1} = u_n - \int_0^t \left[{}_0^C D_\tau^\alpha u_n + R[\tilde{u}_n] + N[\tilde{u}_n] - f(\tau) \right] d\tau \quad (18)$$

However, with the Wu's multiplier:

$$\lambda_W = (-1)^\alpha \frac{(\tau - t)^{\alpha-1}}{\Gamma(\alpha)} \quad (19a)$$

which, in fact, is the kernel of the integration by the Riemann-Liouville integral, we get the Wu's iteration formula, namely:

$$\begin{aligned} u_{n+1} &= u_n + \\ & \int_0^t \left\{ \left[(-1)^\alpha \frac{(\tau - t)^{\alpha-1}}{\Gamma(\alpha)} \right] \times \right. \\ & \left. \times \left[{}_0^C D_\tau^\alpha u_n + R[\tilde{u}_n] + N[\tilde{u}_n] - f(\tau) \right] \right\} d\tau = \quad (19b) \\ & = u_n - {}_0^{RL} I_t^\alpha \left[{}_0^C D_\tau^\alpha u_n + R[\tilde{u}_n] + N[\tilde{u}_n] - f(\tau) \right] \end{aligned}$$

Therefore, the Wu's approach is equivalent to RL integration with $\lambda_H = -1$, as it was suggested mechanistically in this article, but from practical point of view it is more convenient since the Riemann integration can be easily performed by *Maple* or *Mathematica*.

We expect these brief explanations will elucidate the behaviour of the first iterations demonstrated graphically in the previous point and while, in fact some of them were indistinguishable; in fact different combinations led to one and the same results.

Moreover, we may expect that the correct approach in the VIM integration and the Lagrange multiplier determination proposed by Wu would be briefly accepted and successful solutions of problems will be published soon.

VI. Conclusion

The present work formulated a time-fractional Bernoulli equation as a consequent splitting the heat-conduction model into a subdiffusion one (half-time) with derivatives in the Riemann-Liouville sense and application of a non-linear boundary flux condition involving both convective heating and radiative cooling.

The approach is general and can be applied to other combinations of the heat fluxes acting at the solid-fluid interface.

More interesting problem emerging in this work is the correct performance of the successive iterations by the proper identification of the general Lagrange multiplier. The steps done in the analysis of the problem stop at the first iterations only, because before cumbersome calculations it is useful to elucidate which approach is promising. The numerical simulations illustrated by the plots indicate better performance of the Wu's multiplier. This is first estimation, but more information could be taken from solutions of problems having exact closed-form or numerical solutions.

The problem formulated in this work allows the surface temperature evolution of bodies to be calculated in a simpler way avoiding cumbersome solution of the temperature field in the depth of the medium. This would be really a straightforward approach, but some essential steps in the application of VIM, especially the correct Lagrange multiplier identification and consequent application of integration in the Riemann-Liouville since (as it was done in this work) should be further thoroughly analyzed for particular problems.

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