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A Note on an Improved Variational Iteration Method for Nonlinear Equations Arising in Heat Transfer

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Abstract – *An improved variational iteration method for solving nonlinear equations arising in heat transfer is conceived. The main advantage of the present method is that it can enlarge the convergence region of iterative approximate solutions. Also, the present method can be used to solve strong nonlinear problems. Comparison of the result obtained using this method with other existing methods reveals that the present method is more effective and convenient for these types of nonlinear problems.*

***Keywords**: Variational Iteration Method, Heat Transfer*

I. Introduction

The mathematical model describing the temperature distribution in lumped system of combined convection-radiation in a slab made of materials with variable thermal conductivity is given by the following nonlinear boundary value problem:

$$\begin{cases} y''(x) = \varepsilon y^4(x), 0 \le x \le 1 \\ y'(0) = 0, y(1) = 1 \end{cases} \tag{1}$$

where y is dimensionless temperature, x is dimensionless time. Ganji and Rajabi [1],[2] applied homotopy perturbation method to heat radiation equations. Khan [3] developed generalized approximation (GA) method for heat radiation equations.

The variational iteration method (VIM), which was proposed originally by He [4]-[6] has been proved by many authors to be a powerful mathematical tool for addressing various kinds of linear and nonlinear problems [7]-[18]. The reliability of the method and the reduction in the burden of computational work gave this method wider application.

In this paper, we present an improved VIM for solving the problem (1) and obtain its accurate numerical solution.

II. The improved VIM for (1)

The main drawback of the standard VIM is that the sequence of successive approximations of the solution obtained can be rapidly convergent only in a small region, which will greatly restrict the application area of such a method.

To enlarge the convergence region of the sequence of successive approximations obtained, we shall modify the VIM by introducing an auxiliary parameter.

In case of (1) it can be re-written as:

$$y'' - y'' + \gamma\left(y'' - \varepsilon y^4\right) = 0 \tag{2}$$

where γ is an auxiliary parameter and $\gamma \neq 0$, which is used to adjust the convergence region of the following iterative formula.

A correct functional for (2) can be written as:

$$y_{n+1} = y_n + \int_0^x \lambda\left\{y_n'' - \tilde{y}_n'' + \gamma\left[\tilde{y}_n'' - \varepsilon \tilde{y}_n^4\right]\right\} ds \tag{3}$$

where $\tilde{y}_n$ is a restricted variation, i.e. $\delta \tilde{y}_n = 0$; λ is a general Lagrangian multiplier and can be easily identified as $\lambda = s - x$.

Hence, we can obtain the following iteration formula:

$$y_{n+1} = y_n + \gamma\int_0^x (s-x)\left[y_n'' - \varepsilon y_n^4\right] ds \tag{4}$$

According to [6] the approximation (4) can be expressed as:

$$y_{n+1} = y_0 + \int_0^x (s-x)\left[(\gamma-1) y_n'' - \gamma\varepsilon y_n^4\right] ds \tag{5}$$

where y_0 is an initial approximation in the form of $ax+b$ and satisfies the boundary condition of (1).

From the convergence analysis of the VIM, it is easy to see that the smaller the value of $|\gamma|$ is, the wider the convergence region of iterative sequence (5) is. In fact, iterative formula (5) gives us vast freedom of choice. For some strong nonlinear problems, one can choose a relatively small value of $|\gamma|$ (generally less than 1) to obtain a good approximation in a wider region.

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In addition, it should be especially pointed out that when $\gamma = 1$, (5) becomes the standard variational iteration formula.

Theorem 1 Suppose that the iterative sequence $\{y_n(x)\}$ obtained from (5) converge to $y(x)$; then $y(x)$ satisfies (1) and the first boundary condition.

Proof: Taking limits in the iterative formula in (5) and using the integration by parts, it follows that:

$$y(x) = \frac{1}{\gamma}\left[y_0(x) - y(0) - y'(0)\right] - \int_0^x (s-x)\varepsilon y^4 ds \quad (6)$$

Then second order differentiation of both sides with respect to x yields:

$$y''(x) = \varepsilon y^4(x) \quad (7)$$

Obviously, $y(x)$ satisfies (1) and $y'(0) = 0$.

Hence, $y(x)$ satisfies (1) and the first boundary condition, and the proof is complete.

To make $y(x)$ a function satisfying the second boundary condition, we modify (5) in the following way.

With:

$$w(x) = \int_0^x (s-x)\left[(\gamma-1)y_n'' - \gamma\varepsilon y_n^4\right]ds \quad (8)$$

the iteration formula (5) can be expressed as:

$$y_{n+1} = y_0 + w(x) - w(1) \quad (9)$$

Clearly we have $y_n(1) = 1$. From the expression (9) and under the conditions imposed by Theorem 1 it follows that $y(x)$ satisfies the model (1) and its two boundary conditions. Therefore, $y(x)$ is the solution of (1).

III. Results and Discussion

In this section, we apply the method presented in Section II to (1). Numerical results show that the present method is very effective.

Since, the exact solution to the model (1) cannot be obtained; the results obtained by using the present method (PRM) are compared with results obtained by the software Maple, GA and the homotopy perturbation method (HPM).

Using the PRM and taking $\gamma = 0.5$ for $\varepsilon = 0.6, 2.0$, the numerical results compared with other methods are shown in Tables I, II.

From the numerical results, it is shown that, in contrast with other existing methods like GA and HPM, the present method can provide more accurate approximate solutions to heat radiation equations.

TABLE I
NUMERICAL RESULTS FOR $\varepsilon = 0.6$

x	Maple	PRM	GA	HPM
0	0.834542	0.834543	0.963536	0.640000
0.2	0.840390	0.840391	0.964009	0.652096
0.4	0.858269	0.858269	0.965742	0.689536
0.6	0.889247	0.889248	0.969893	0.755776
0.8	0.935346	0.935346	0.979233	0.866576

TABLE II
NUMERICAL RESULTS FOR $\varepsilon = 2.0$

x	Maple	PRM	GA	HPM
0	0.694318	0.694318	0.968771	-0.666667
0.2	0.703698	0.703698	0.968804	-0.625600
0.4	0.732894	0.732894	0.969008	-0.489600
0.6	0.785488	0.785489	0.970024	-0.220267
0.8	0.869161	0.869164	0.975059	-0.246400

IV. Conclusion

In this paper, an improved VIM is presented for solving nonlinear equations arising heat transfer. The present method can give a more accurate approximation in a larger region. Therefore, the improvement overcomes the restriction of the application area of the VIM, and then expands its scope of application. The present method can be applied to other strong nonlinear problems.

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