International Review of Chemical Engineering (I.RE.CH.E.), Vol. 3, N. 6
November 2011
Special Section on ODI Collection

Stokes Flows of a Newtonian Fluid with Fractional Derivatives and Slip at the Wall

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Abstract – *Stokes flows of a Newtonian fluid with fractional derivatives produced by the motion of a flat plate are analyzed under the slip condition at boundary. The plate motion is assumed to have a translation in its plane with a given velocity and the relative velocity between the velocity of the fluid at the wall and the speed of the wall is assumed to be proportional to the shear rate at the wall. The exact expressions for the velocity and the shear stress are determined by means of the Laplace transform. The velocity fields corresponding to both cases with slip and non-slip conditions, for fractional Newtonian and Newtonian fluids are obtained. The particular case, namely sine oscillations of the wall is studied. Results for fractional Newtonian fluids are compared with those of viscous Newtonian fluids in both cases of the flow with slip and non-slip conditions. In addition the influence of the slip coefficient on the relative velocity is studied.*

***Keywords**: Newtonian Fluid, Fractional Derivative, Velocity Field*

I. Introduction

The assumption that a liquid adheres to a solid boundary so called "non-slip" boundary condition-is one of the principal rules of the Navier Stokes theory. More experiments are in favor of the non-slip boundary condition for a large class of flows. An interesting discussion regarding to acceptance of the non-slip condition can be found in [1].

Even if this assumption has proved to be successful for a great variety of flows, it has been found to be inadequate in several situations such as: the mechanics of thin fluids, problems involving multiple interfaces, flows in micro-channels or in wavy tubes, or flows of polymeric liquids with high molecular weight. Navier [2] proposed a slip boundary condition which states that the velocity of the fluid at the wall is linearly proportional to the shear stress at the wall.

For describing the slip that occurs at solid boundaries, a large number of models have been proposed. Many of them can be found in [3].

Generally, the slip is assumed to depend on the shear stress at the wall. However, more experiments suggest that the slip velocity also depends on the normal stress [3].

The flow of Newtonian or non-Newtonian fluids induced by a motion of a plate is called as stokes problem if the fluid is bounded only by the moving plate and Couette flow, if the fluid is bounded by two parallel plates. Solutions for some Stokes flows due to a moving plate that satisfy non slip conditions can be found in references [4]-[7]. One of the early studies of the slip at the boundary was undertaken by Mooney [8].

Some non-Newtonian fluids such as polymer melts, often exhibit macroscopic wall slip, which generally described by a nonlinear relation between the wall slip velocity and the fraction at the wall [9].

Recently, the fractional calculus has encountered much success in the disruption of complex dynamics. In particular, it has been proved to be a valuable tool for handling viscoelastic properties [10], [11]. Some interesting results regarding to the flows of Newtonian or non-Newtonian fluids with fractional derivatives can be found in [12]-[14].

In this paper, Stokes flows of a Newtonian fluid with fractional derivatives, produced by the motion of a flat plate are analyzed under the slip boundary conditions assumption between the wall and the fluid. The motion of the wall is a rectilinear translation in its plane with the velocity $u_w(t) = f(t)$. Exact expressions for the velocity and shear stress are determined by means of the Laplace transform. Expressions for the relative velocity are also determined and the solutions corresponding to flows with non-slip at the boundary are presented. The particular case, namely sine oscillations of the wall is studied. Some relevant properties of the velocity and comparisons between solutions with slip and non-slip at the boundary are presented by using graphical illustrations generated by the software MathCAD.

II. Problem Formulation and Solution

Consider a plane wall situated in the xz-plane of a Cartesian coordinate system with the positive y-axis in the upward direction. Let an incompressible,

Manuscript received and revised October 2011, accepted November 2011

homogeneous Newtonian fluid with fractional derivatives fill the region $y \geq 0$. Initially, both the fluid and plate are at rest, At time $t = 0^+$ the fluid is set in motion by the plate which begins to translate along the x-axis with the velocity $u_w(t) = f(t)$, where $f(t)$ is a piecewise continuous function defined on $(0, \infty]$ with $f(0) = 0$. Also, we suppose that the Laplace transform of the function f exists. In the case of a parallel flow along the x-axis, the velocity vector is $\mathbf{V} = (u(y,t), 0, 0)$ while the constitutive relation and the governing equations are given by [12], [15]:

$$\tau(y,t) = \mu \frac{\partial u(y,t)}{\partial y}, \quad y, t > 0 \tag{1}$$

$$^{c}D_t^{\alpha} u(y,t) = \nu_1 \frac{\partial^2 u(y,t)}{\partial y^2}, \quad y, t > 0 \tag{2}$$

where $\tau(y,t) = S_{xy}(y,t)$ is one of the non-zero components of the extra-stress tensor, μ is the dynamic viscosity, $\nu_1 \left[m^2 / s^{\alpha} \right], \nu_1 > 0$, is the material coefficient and $^{c}D_t^{\alpha}$ is the Caputo fractional derivative operator defined by [10], [11]:

$$^{c}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau, \quad 0 < \alpha < 1 \tag{3}$$

Γ being the Gamma function.

For $\alpha = 1$, $^{c}D_t^{1} f(t) = \dfrac{df(y,t)}{dt}$, $\nu_1 = \nu = \mu / \rho$ is the kinematic velocity, ρ being the density of the fluid and Eq. (2) governs the flow of Newtonian fluids. In this paper we consider the existence of slip at the wall and assume that the relative velocity between $u(0,t)$- the velocity of the fluid at the wall and the speed of the wall is proportional to the shear rate at the wall [16]. The boundary and initial conditions are:

$$u(0,t) - \beta \frac{\partial u(0,t)}{\partial y} = u_w(t) \tag{4}$$

$$u(y,t) \to 0 \;\; as \;\; y \to \infty \tag{5}$$

$$u(y,0) = 0 \tag{6}$$

where β is the slip coefficient. The Laplace transform method is used to solve Eq. (2) with initial-boundary conditions (4)-(6)

II.1. Velocity field

By applying the temporal Laplace transform [12] to Eqs. (2), (4), (5) and using the initial condition (6) we obtain the following problem:

$$\frac{\partial^2 \bar{u}(y,q)}{\partial y^2} = \frac{1}{\nu_1} q^{\alpha} \bar{u}(y,q) \tag{7}$$

$$\bar{u}(0,q) - \beta \frac{\partial \bar{u}(0,q)}{\partial y} = F(q) \tag{8}$$

$$\bar{u}(y,q) \to 0 \;\; as \;\; y \to \infty \tag{9}$$

where $\bar{u}(y,q)$ and $F(q)$ are the Laplace transforms of functions $u(y,t)$ and $f(t)$ respectively.

The solution of the differential equation (7) with conditions (8) and (9) is given by:

$$\begin{aligned} \bar{u}(y,q) &= \frac{F(q)}{1 + \frac{\beta}{\sqrt{\nu_1}} q^{\alpha/2}} e^{-\frac{y}{\sqrt{\nu_1}} q^{\alpha/2}} \\ &= F(q) \frac{\sqrt{\nu_1} q^{-\alpha/2}}{\beta \left(q^{\alpha/2} + \frac{\sqrt{\nu_1}}{\beta} \right)} q^{\alpha/2} e^{-\frac{y}{\sqrt{\nu_1}} q^{\alpha/2}} \end{aligned} \tag{10}$$

Taking the inverse Laplace transform [17], [18] in Eq. (10), using (A-1) and (A-2) from the Appendix and the convolution theorem we obtain the following *exact expression for the velocity corresponding to the flow of a fractional Newtonian fluid with slip at the boundary*:

$$\begin{aligned} u(y,t) = & \frac{2\sqrt{\nu_1}}{\alpha\beta} \sum_{n=0}^{\infty} \frac{\left(-\frac{y}{\sqrt{\nu_1}} \right)^n}{(n+1)!\,\Gamma\left(\frac{\alpha(n+1)}{2} \right)} \times \\ & \times \int_0^t \int_0^s \int_0^{\infty} f(t-s) \sigma^{\alpha-1} E_{\frac{\alpha}{2},\alpha}\left(-\frac{1}{\beta} \sqrt{\nu \sigma^{\alpha}} \right) z^{\frac{\alpha(n+1)}{2}} \times \\ & \times J_0\left(2\sqrt{(s-\sigma)z} \right) dz\, d\sigma\, ds \end{aligned} \tag{11}$$

where $E_{\alpha,\beta}(\cdot)$ is the Mittag-Leffler function [18] and $J_p(\cdot)$ denotes the Bessel function of the first kind of order p. For the flow of a fractional Newtonian fluid with non-slip boundary condition, that is $\beta = 0, 0 < \alpha < 1$, Eq. (10) becomes:

$$\bar{u}(y,q) = F(q) \frac{1}{q^{\alpha/2}} q^{\alpha/2} e^{-\frac{y}{\sqrt{\nu_1}} q^{\alpha/2}} \tag{12}$$

and the (y, t) -domain solution is given by:

$$u(y,t) = \frac{2}{\alpha\Gamma\left(\frac{\alpha}{2}\right)} \sum_{n=0}^{\infty} \frac{\left(-\frac{y}{\sqrt{\nu_1}}\right)^n}{(n+1)!\,\Gamma\left(\frac{\alpha(n+1)}{2}\right)} \times \int_0^t \int_0^s \int_0^{\infty} f(t-s)(s-\sigma)^{\frac{\alpha-2}{2}} z^{\frac{\alpha(n+1)}{2}} \times J_0\left(2\sqrt{\sigma z}\right) dz\, d\sigma\, ds \tag{13}$$

The velocity field corresponding to the flow of a Newtonian fluid with slip at the boundary, that is for $\alpha = 1, \beta \neq 0$ is given by:

$$u(y,t) = \int_0^t f(t-s) \times \left[\frac{\sqrt{\nu}}{\beta\sqrt{\pi s}} exp\left(-\frac{y^2}{4\nu s}\right) + -\frac{\nu}{\beta^2} exp\left(-\frac{\beta y + \nu s}{\beta^2}\right) Erfc\left(\frac{y}{2\sqrt{\nu s}} + \frac{\sqrt{\nu s}}{\beta}\right) \right] ds \tag{14}$$

while the velocity corresponding to the flow of a Newtonian fluid with non-slip condition, that is $\alpha = 1, \beta = 0$ is:

$$u(y,t) = \frac{y}{2\sqrt{\nu\pi}} \int_0^t \frac{f(t-s)}{s\sqrt{s}} exp\left(-\frac{y^2}{4\nu s}\right) ds \tag{15}$$

The relative velocity between the velocity of the fluid at the wall and the wall itself for a fractional Newtonian fluid is given by:

$$\begin{aligned} u_{FNrel}(t) &= u\left(0^+, t\right) - f(t) = \\ &= L^{-1}\left\{ \lim_{y \to 0^+} \frac{F(q)}{1 + \frac{\beta}{\sqrt{\nu_1}} q^{\alpha/2}} e^{-\frac{y}{\sqrt{\nu_1}} q^{\alpha/2}} \right\} - f(t) = \\ &= L^{-1}\left\{ \frac{\sqrt{\nu_1}}{\beta} \frac{F(q)}{\left(q^{\alpha/2} + \frac{\sqrt{\nu_1}}{\beta}\right)} \right\} - f(t) = \\ &= \frac{\sqrt{\nu_1}}{\beta} \int_0^t f(t-s) s^{\frac{\alpha}{2}-1} E_{\frac{\alpha}{2},\frac{\alpha}{2}}\left(-\frac{1}{\beta}\sqrt{\nu_1 s^{\alpha}}\right) ds - f(t) \end{aligned} \tag{16}$$

and, for a Newtonian fluid is:

$$\begin{aligned} u_{Nrel}(t) &= L^{-1}\left\{ \lim_{y \to 0^+} \frac{F(q)}{1 + \frac{\beta}{\sqrt{\nu}}\sqrt{q}} e^{-\frac{y}{\sqrt{\nu}}\sqrt{q}} \right\} - f(t) = \\ &= L^{-1}\left\{ \frac{\sqrt{\nu}}{\beta} \frac{F(q)}{\left(\sqrt{q} + \frac{\sqrt{\nu}}{\beta}\right)} \right\} - f(t) = \\ &= \frac{\sqrt{\nu}}{\beta} \int_0^t \frac{f(t-s)}{\sqrt{s}} E_{\frac{1}{2},\frac{1}{2}}\left(-\frac{1}{\beta}\sqrt{\nu s}\right) ds = \\ &= \frac{\sqrt{\nu}}{\beta} \int_0^t f(t-s) \left[\frac{1}{\sqrt{\pi s}} - \frac{\sqrt{\nu}}{\beta} e^{\frac{\nu s}{\beta^2}} Erfc\left(\frac{\sqrt{\nu s}}{\beta^2}\right) \right] ds \end{aligned} \tag{17}$$

II.2. Shear Stress

The shear stress, determined by using Eqs. (1), (12)-(14) is given by:

a) *The Newtonian fluids with fractional derivatives and slip at the wall*

$$\tau(y,t) = \frac{2\sqrt{\nu_1}}{\alpha\beta} \frac{\mu}{y} \sum_{n=1}^{\infty} \frac{n\left(-\frac{y}{\sqrt{\nu_1}}\right)^n}{(n+1)!\,\Gamma\left(\frac{\alpha(n+1)}{2}\right)} \times \int_0^t \int_0^s \int_0^{\infty} \left[\begin{array}{l} f(t-s)\sigma^{\alpha-1} \cdot \\ \cdot E_{\frac{\alpha}{2},\alpha}\left(-\frac{1}{\beta}\sqrt{\nu_1 \sigma^{\alpha}}\right) z^{\frac{\alpha(n+1)}{2}} \cdot \\ \cdot J_0\left(2\sqrt{(s-\sigma)z}\right) \end{array} \right] dz\, d\sigma\, ds \tag{18}$$

b) *Newtonian fluid with fractional derivatives and non-slip at the wall*

$$\tau(y,t) = \frac{2\mu}{\alpha\Gamma\left(\frac{\alpha}{2}\right) y} \sum_{n=1}^{\infty} \frac{n\left(-\frac{y}{\sqrt{\nu_1}}\right)^n}{(n+1)!\,\Gamma\left(\frac{\alpha(n+1)}{2}\right)} \times \int_0^t \int_0^s \int_0^{\infty} \left[\begin{array}{l} f(t-s)(s-\sigma)^{\frac{\alpha-2}{2}} \cdot \\ \cdot z^{\frac{\alpha(n+1)}{2}} J_0\left(2\sqrt{\sigma z}\right) \end{array} \right] dz\, d\sigma\, ds \tag{19}$$

c) *Newtonian fluid with slip at the boundary*

$$\tau(y,t)=\mu\int_0^t f(t-s)\times \left[\begin{array}{l}\dfrac{2\nu s-\beta y}{2\beta s\sqrt{\nu\pi s}}exp\left(-\dfrac{y^2}{4\nu s}\right)-\\ \dfrac{\nu}{\beta^3}exp\left(\dfrac{\beta y+\nu s}{\beta^2}\right)Erfc\left(\dfrac{y}{2\sqrt{\nu s}}+\dfrac{\sqrt{\nu s}}{\beta}\right)\end{array}\right]ds \qquad (20)$$

d) *Newtonian fluid with non-slip condition*

$$\tau(y,t)=\frac{\mu}{2\sqrt{\nu\pi}}\int_0^t \frac{f(t-s)}{s\sqrt{s}}exp\left(-\frac{y^2}{4\nu s}\right)ds+ -\frac{\mu y^2}{4\nu\sqrt{\nu\pi}}\int_0^t \frac{f(t-s)}{s^2\sqrt{s}}exp\left(-\frac{y^2}{4\nu s}\right)ds \qquad (21)$$

III. Flows Due to Sine Oscillations of the Wall

The velocity fields corresponding to this type of motion are given by Eqs. (11)-(14) in which function $f(t-s)$ is replaced by $V_0 \sin\omega(t-s)$, ω being the frequency of oscillations. By using the graphical illustrations, generated with the software MathCAD, we discuss some relevant physical aspects of the flow. In these figures we use $V_0 = 0.5\ (\text{m/s})$, $\nu_1 = 0.1655(0.2)^{1-\alpha}\ (\text{m}^2/\text{s}^\alpha)$, $0<\alpha\le 1$, $\nu = 0.1655\ \text{m}^2/\text{s}$, $\omega=\pi/6$ and the following abbreviations:

I. NF_s -fractional Newtonian fluid with slip at the boundary,
II. NF -fractional Newtonian fluid with non-slip condition at the boundary,
III. N_s -Newtonian fluid with slip at the boundary, N - Newtonian fluid with non-slip at the boundary,

In Fig. 1 we plotted the velocity $u(y,t)$ verses t for $y\in\{0.1, 0.2, 0.3\}$. For comparison, we have plotted these functions corresponding to fractional Newtonian and Newtonian fluids with slip boundary and with non-slip boundary conditions. It is clear that the absolute values of the velocity decrease if y or t increases. For both, fractional Newtonian and Newtonian fluids the velocity is larger in the case of non-slip condition than in the case of slip at the boundary. The fractional Newtonian fluid is slower than the Newtonian fluid.

In Fig. 2 we drew diagrams of velocity $u(y,t)$ verses t, for three different values of the fractional coefficients α. The curves correspond to fractional Newtonian fluids with slip at the wall. For comparison the curves corresponding to the Newtonian fluid with slip at the wall is also plotted.

From these figures, it clearly results that the velocity $u(y,t)$, in absolute terms, increases for increasing α. For $\alpha\to 1$ the diagrams of velocity tend to the diagram corresponding to a Newtonian fluid with slip at the boundary.

In Fig. 3 we have plotted the relative velocity verses t for three different values of the slip coefficient β, for both fractional Newtonian and Newtonian fluids. The relative velocity in absolute terms, is an increasing function of β.

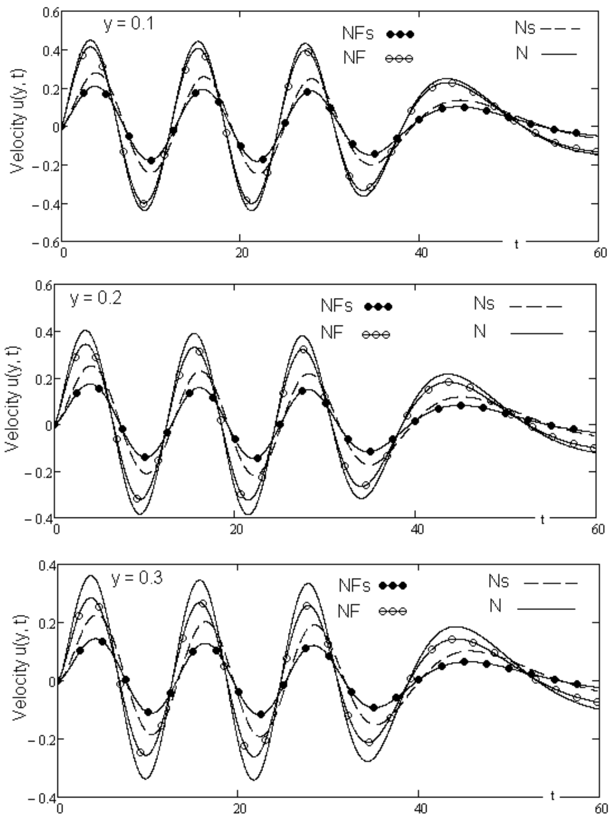

Fig. 1. Velocity u(y, t) versus t for V_0=0.5, α = 0.7, β = 0.5, ν =0.1655, $\nu_1 = 0.1655\cdot(0.2)^{1-\alpha}$, ω = π/6 and different values of y

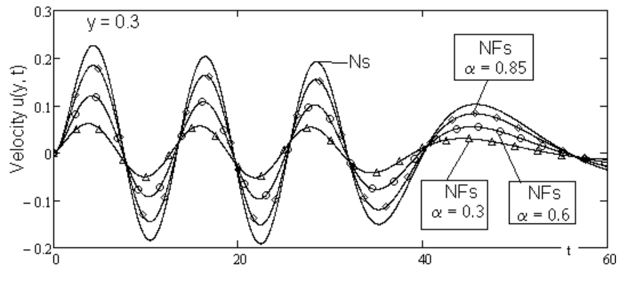

Fig. 2. velocity u(y, t) verses t for V_0=0.5, β = 0.5, ν =0.1655, $\nu_1 = 0.1655\cdot(0.2)^{1-\alpha}$, ω = π/6 and different values of α

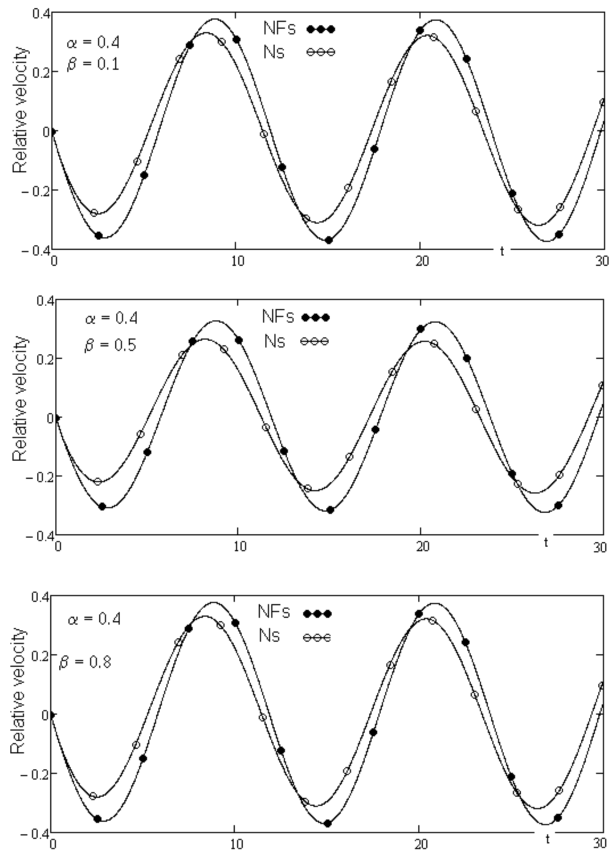

Fig. 3. Relative velocity u(t) for V_0=0.5, ν = 0.1655, $\nu_1 = 0.1655 \cdot (0.2)^{1-\alpha}$, ω = π/6 and different values of β

IV. Conclusion

Stokes flows of a fractional Newtonian fluid were analyzed under the slip condition between the fluid and the wall. The motion of the wall was assumed to be a rectilinear translation in its plane. A particular case, namely sine oscillations was considered. The relative velocity between the velocity of the fluid at the wall and the wall was assumed to be proportional to the shear rate at the wall. The exact expressions for the velocity $u(y,t)$ and shear stress $\tau(y,t)$ have been determined by means of the Laplace transform. For a complete study and possible comparisons, we present velocity fields corresponding to both flows with slip and non-slip conditions for fractional Newtonian and Newtonian fluids. The expressions for the relative velocity have also been determined. For sine oscillations of the plate, the velocity corresponding to flows with slip condition is smaller than that for flows with non-slip conditions and fractional Newtonian fluids are slower than Newtonian fluids (See Fig. 1). The velocity in absolute terms increases for increasing fractional coefficient α, (Fig. 2); the relative velocity in absolute terms is an increasing function of the slip coefficient β. The software MathCAD 14.0 was used for numerical calculations and to generate the diagrams presented in this paper.

Appendix

$$L\left\{t^{\delta-1}E_{\gamma,\delta}\left(-at^{\gamma}\right)\right\} = \frac{q^{\gamma-\delta}}{q^{\gamma}+a}, \qquad \text{(A-1)}$$

$$Re(q) > |a|^{\frac{1}{\alpha}}, \quad \gamma, \delta > 0$$

$$L\left\{q^{b}e^{aq^{b}}\right\} = \frac{1}{b}\sum_{n=0}^{\infty}\frac{a^{n}}{(n+1)!\,\Gamma\left[b(n+1)\right]}\times$$

$$\times\int_{0}^{\infty} z^{b(n+1)}J_0\left(2\sqrt{tz}\right)dz, \quad b>0,\, a\neq 0 \qquad \text{(A-2)}$$

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